Daniela Gancheva Marinova

SPECIAL TOPICS IN CONTROL THEORY

CONTROLLABILITY AND OBSERVABILITY

$$\dot{x} = Ax + Bu$$

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SPECIAL TOPICS IN CONTROL THEORY (CONTROLLABILITY AND OBSERVABILITY)

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Chapter 1

The Kalman criterion I

1.1 Introduction

In these lectures we study, in particular, the issue of controllability of control systems.

In this chapter we consider the Kalman criterion of complete controllability. This notion is defined as follows:

Definition 1.1.1 *Suppose we are given a general control system*

$$\dot{x} = f(x, u) \tag{1.1.1}$$

where u is somehow constrained. Any solution defines an admissible curve. *The system (1.1.1) is said to be* completely controllable *iff (if and only if) any two points x', x'' of the phase space can be connected by an admissible curve.*

We use widely the closely related notion of attainable (reachable) set

Definition 1.1.2 *Let M be a subset of the phase space $t \geq 0$, and $\mathcal{D}(t; M) = \mathcal{D}(t)(M)$ is the set of ends $x(t) = x^u(t)$ of all admissible trajectories such that $x(0) \in M$. The set $\mathcal{D}(t; M) = \mathcal{D}(t)(M)$ is said to be attainable (reachable) set of (1.1.1)*

Then it is obvious that the system (1.1.1) is completely controllable iff

$$\mathcal{D} = \mathcal{D}(M) = \bigcup_{t>0} \mathcal{D}(t)(M) \tag{1.1.2}$$

coincides with the entire phase space for any initial set M.

1.2 Linear systems with unconstrained control

These are the control system of the form

$$\dot{x} = Ax + Bu \tag{1.2.1}$$

Here, the phase space $V = \mathbf{R}^n$ is a finite dimensional vector space, and the space of control vectors $U = \mathbf{R}^m$ is another finite dimensional space. We consider primarily the time invariant case, where A and B are constant matrices.

Our goal is to find a criterion of controllability for the system. There is a general important question: "How to specify the admissible control functions $t \mapsto u(t)$?" In the case at hands the choice is not very relevant, for very different functional spaces give us the same attainable sets. For definiteness we assume that the admissible controls are the locally-integrable functions: i.e. u is a measurable vector-function of time such that

$$\int_0^T |u(t)|\, dt < \infty. \tag{1.2.2}$$

We would get similar results taking, say infinitely differentiable controls, or distributional controls. In the last case we have to be careful, because the solution x of $\dot{x} = Ax + Bu$, where u is a distribution supported on $[0, \infty)$, is a distribution, not a (measurable) function, and we cannot speak directly of "the ends of trajectories $x(t)$". However, for some subspaces of distributions, like measures, there is no such a problem.

The crucial question we are dealing with is this: "How to describe the attainable set $\mathcal{D} = \bigcup_t \mathcal{D}(t)(\{0\})$?" (the initial set M is the origin)

So we start with the description.
The first main properties of $\mathcal{E} = \mathcal{E}(t) = \mathcal{D}(t)(\{0\})$ are these:

1. $\mathcal{E}$ is a vector space

2. $\mathcal{E}(t') \subset \mathcal{E}(t'')$ if $t' \leq t''$

Both properties are almost obvious. To prove first, we note that if $x' = x^{u'}(t)$, $x'' = x^{u''}(t)$, then $x' + x'' = x^{u'+u''}(t)$, and that $\lambda x^u(t) = x^{\lambda u}(t)$, where λ is a constant.

To prove second, we have to give a recipe of procrastination: Suppose $x = x^u(t)$ and $s = t + \delta$, where $\delta > 0$. Then we define a new control

$$v(\tau) = \begin{cases} 0, & \text{if } \tau < \delta \\ u(\tau - \delta), & \text{otherwise.} \end{cases} \tag{1.2.3}$$

In other words we stay at the origin in the interval $[0, \delta]$ of time, and then apply the shifted control u. One can see easily that $x^u(t) = x^v(s)$. This proves that $\mathcal{E}(t) \subset \mathcal{E}(t + \delta)$.

In fact, the second property has nothing to do with linearity: the same applies to any autonomous control system and initial set M consisting of *equilibrium* points for the zero control.

It follows immediately from 1, 2 that $\mathcal{D}$ is a vector subspace of $V = \mathbf{R}^n$. Moreover, every admissible curve, starting from $\mathcal{D}$ remains in it forever. The last property can be expressed as follows:

3. If $x \in \mathcal{D}$, then $Ax + Bu \in \mathcal{D}$ for any $u \in U$

(the tangent vector to admissible trajectory $Ax + Bu$ in the vector space $\mathcal{D}$ at point x belongs to $\mathcal{D}$)

This can be restated as follows:

4. The space $\mathcal{D}$ contains $W = BU$ and is invariant under the matrix A.

On the other hand it is clear that any vector subspace of V that contains $W = BU$ and is invariant under the matrix A contains any admissible curve coming from the origin. We conclude, that

5. The space $\mathcal{D}$ is minimal vector subspace of V such that it contains BU and is invariant under the matrix A.

Therefore, if our system (7.3.1) is completely controllable, then

6. any vector subspace of V that contains BU and is invariant under A coincides with V.

Conversely, if the latter condition holds, our system is completely controllable.

Indeed, the condition guarantee that we can get to any point from the origin along an admissible path, and it remains to show that we can get to the origin from any point. This, however, corresponds to the motion from

the origin to the point with time reversed, which, in turn, corresponds to the substitution $A \to -A$ in governing equation (7.3.1). However, under condition 6 any vector subspace of V that contains BU and is invariant under $-A$ coincides with V.

Thus, we proved the following

Theorem 1.2.1 *The system (7.3.1) is completely controllable iff any vector subspace of $V = \mathbf{R}^n$ that contains BU and is invariant under A coincides with V.*

The original Kalman controllability condition is a slight improvement of Theorem 1.2.1:

Theorem 1.2.2 *Define the* Kalman matrix *of size $n \times nm$*

$$K = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{1.2.4}$$

The system (7.3.1) is completely controllable iff the rang $\operatorname{rk} K$ *of the Kalman matrix is the dimension n of the phase space (maximal possible).*

Proof. The minimal subspace $\mathcal{D}$ of $V = \mathbf{R}^n$ that contains BU and is invariant under A consists of vectors of the form

$$\mathcal{D} = \left\{ \sum_{i \geq 0} A^i B u_i, \quad u_i \in U \right\} \tag{1.2.5}$$

where the sum is finite. On the other hand, the range of the Kalman matrix consists of vectors of the form

$$KU^n = \left\{ \sum_{i \geq 0}^{n-1} A^i B u_i, \quad u_i \in U \right\} \tag{1.2.6}$$

It remains to note that $A^n = \sum_{i \geq 0}^{n-1} a_i A^i$ by the Caley–Hamilton theorem, which proves that $\mathcal{D} = KU^n$. ▲

We remind that the Caley–Hamilton theorem says, that if

$$p(X) = \det(X - A)$$

is the characteristic polynomial of a matrix A, then $p(A) = 0$.

One important property of linear control systems (7.3.1) remains unsettled by now: In fact, the attainable set $\mathcal{D} = \bigcup_{t>0} \mathcal{D}(t)(\{0\})$ coincides with

$\mathcal{D}(T)(\{0\})$ for any $T > 0$. We will establish this later on. However, it is easy to prove the said property by methods of this lecture, and the reader might do this as an exercise.
Hint: For any $\delta > 0$ consider $\mathcal{D}_\delta = \bigcup_{0<t<\delta} \mathcal{D}(t)(\{0\})$ instead of

$$\mathcal{D} = \bigcup_t \mathcal{D}(t)(\{0\})$$

and follow the steps **1-6.**

Chapter 2

The Kalman criterion II.

In this chapter we discus the attainable sets of linear systems with bouded and unbounded control.

The Kalman controllability condition is generic, so that a linear control system

$$\dot{x} = Ax + Bu \tag{2.0.1}$$

in general position should be completely controllable. Another general remark is that the complexification does not affect the controllability: The (7.3.2) is controllable iff the complexified system

$$\dot{z} = A_{\mathbf{C}} z + B_{\mathbf{C}} w \tag{2.0.2}$$

is controllable. Here, the phase and control spaces $V_{\mathbf{C}} = \mathbf{C}^n$ and $U_{\mathbf{C}} = \mathbf{C}^m$ are complexifications of V and U, and the same holds for the matrices involved.

2.1 A general theorem

Consider the root decomposition $V_{\mathbf{C}} = \bigoplus V_\lambda$, where

$$V_\lambda = \{x \in V_{\mathbf{C}} : (A - \lambda)^k x = 0 \text{ for some positive integer } k\}, \tag{2.1.1}$$

and the corresponding projectors $p_\lambda : V_{\mathbf{C}} \to V_\lambda$. The existence of the root decomposition for any operator $A : V \to V$ is one of the basic results of linear algebra. Define a linear control system in the space V_λ by

$$\dot{x}_\lambda = A_{\mathbf{C}} x_\lambda + B_\lambda w, \tag{2.1.2}$$

where $B_\lambda = p_\lambda B_{\mathbf{C}}$.

Theorem 2.1.1 *The system (7.3.2) is completely controllable iff all the systems (2.1.2) are so.*

Proof. Denote by $\mathcal{D}$ the reachable set from the origin for system (7.3.2), and by $\mathcal{D}_\lambda$ the corresponding set for system (2.1.2). We have to prove that the statement $\mathcal{D} = V$ is equivalent to $\mathcal{D}_\lambda = V_\lambda$ for all λ. Indeed, $\mathcal{D}_{\mathbf{C}}$ is A invariant, and, thus, has the root decomposition

$$\mathcal{D}_{\mathbf{C}} = \bigoplus \mathcal{D}_{\mathbf{C}} \cap V_\lambda = \bigoplus p_\lambda \mathcal{D}_{\mathbf{C}}.$$

Each space $p_\lambda \mathcal{D}_{\mathbf{C}} = \mathcal{D}_{\mathbf{C}} \cap V_\lambda$ contains the image $B_\lambda U_{\mathbf{C}}$ of $U_{\mathbf{C}}$ and is A-invariant. Therefore, it contains $\mathcal{D}_\lambda$. On the other hand, if $x(t)$ is an admissible curve of (7.3.2) its projection $x_\lambda(t) = p_\lambda x(t)$ satisfies (2.1.2) which implies that $\mathcal{D}_\lambda$ contains $p_\lambda \mathcal{D}_{\mathbf{C}}$. We conclude, that $\mathcal{D}_\lambda = p_\lambda \mathcal{D}_{\mathbf{C}}$, and $\mathcal{D}_{\mathbf{C}} = \bigoplus \mathcal{D}_\lambda$. The statement $\mathcal{D} = V$ is equivalent to $\mathcal{D}_{\mathbf{C}} = V_{\mathbf{C}}$, which, in turn, just means that $\bigoplus \mathcal{D}_\lambda = \bigoplus V_\lambda$. The latter is clearly equivalent to $\mathcal{D}_\lambda = V_\lambda$ for all λ. ▲

We state a corollary of the proved theorem:

Theorem 2.1.2 *Suppose that A has distinct eigenvalues $\{\lambda\}$, the controls are scalar, i.e., B is a vector. Then (as is well-known) the matrix A is semisimple (diagonalizable), and the vector B has the decomposition $B = \sum B_\lambda e_\lambda$, where e_λ are eigenvectors of A. Then, the system (7.3.2) is completely controllable iff $B_\lambda \neq 0$ for all λ.*

Proof is left to the listener as an exercise. ▲

The same conclusion can be reached by explicit calculations with the Kalman matrix. The Kalman matrix in the basis e_λ takes the form

$$\begin{pmatrix} B_{\lambda_1} & B_{\lambda_1}\lambda_1 & \dots & B_{\lambda_1}\lambda_1^{n-1} \\ B_{\lambda_2} & B_{\lambda_2}\lambda_2 & \dots & B_{\lambda_2}\lambda_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ B_{\lambda_n} & B_{\lambda_n}\lambda_n & \dots & B_{\lambda_n}\lambda_n^{n-1} \end{pmatrix} \tag{2.1.3}$$

and its determinant is equal to

$$\left(\prod_\lambda B_\lambda\right) \det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix} \tag{2.1.4}$$

Since all λ_i are distinct, the latter Wandermonde determinant $\neq 0$. By assumption $\prod_\lambda B_\lambda \neq 0$, and the Kalman matrix is nonsingular.

The fact that the Wandermonde determinant $\neq 0$ is well-known and follows, say, from the explicit formula

$$\det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix} = \prod_{i<j} (\lambda_i - \lambda_j) \tag{2.1.5}$$

Even more generally, consider the system (7.3.2) with the scalar control, where the Jordan normal form of the matrix A is determined by the set $\{n_\lambda\}$ of integer vectors, parametrized by the spectrum of A. The vector $n_\lambda = (n_\lambda^1, \dots, n_\lambda^{k_\lambda})$ consist of sizes of the Jordan cells corresponding to the eigenvalue λ. Denote by p_λ^i the spectral projector, corresponding to the Jordan cell of size n_λ^i. This is defined uniquely if all these sizes are distinct (for a fixed λ).

Theorem 2.1.3 *The system (7.3.2) with the scalar control is completely controllable iff each of the vectors n_λ has all components distinct, and $(A - \lambda)^{n_\lambda^i - 1} p_\lambda^i B \neq 0$ for each i and λ.*

Proof, which follows the same line as that of Theorem 2.1.2 is left to the listener as an exercise.

Corollary 2.1.4 *The system*

$$\begin{aligned} &\ddot{x}_i + \omega_i x_i = u, \quad i = 1, \dots, N \\ &\ddot{x} = u \end{aligned} \tag{2.1.6}$$

which describes the set of pendulums, attached to a cart moving under a force applied in a fixed direction, is completely controllable iff all $\omega_i > 0$, $i = 1, \dots, N$ are distinct and not equal to zero.

Similarly, we have a simpler

Corollary 2.1.5 *The system*

$$\ddot{x}_i + \omega_i x_i = u, \quad i = 1, \dots, N \tag{2.1.7}$$

which describes the set of pendulums under a force applied in a fixed direction, is completely controllable iff all $\omega_i > 0$, $i = 1, \dots, N$ are distinct.

2.2 Kalman controls

We turn to the issue of actual design of controls bringing one state of the linear system (7.3.2) to another. We already know that it is sufficient, in principle, to come from the origin to the given point $x \in V$.

We need the Cauchy formula for a solution of (7.3.2):

$$x(T) = x^u(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)}Bu(s)\,ds \tag{2.2.1}$$

In particular, if $x(0) = 0$ then

$$x(T) = x^u(T) = \int_0^T e^{A(T-s)}Bu(s)\,ds \tag{2.2.2}$$

Following Kalman we fix any $T > 0$, and consider controls of the form

$$u_\xi(s) = B^* e^{A^*(T-s)}\xi \tag{2.2.3}$$

where * stands for the transposition, and $\xi \in V^* = \mathbf{R}^n$.

Theorem 2.2.1 *If the system (7.3.2) is completely controllable, then one can get from the origin to a given point x at any given time $T > 0$ by using the (unique) control of the form (2.2.3).*

Proof. Consider the operator $P : V^* \to V$ given by

$$P\xi = \int_0^T e^{A(T-s)}Bu_\xi(s)\,ds \tag{2.2.4}$$

The conclusion of the theorem is equivalent to the fact that P is an invertible matrix. To prove this, we note that

$$(P\xi, \xi) = \int_0^T |B^* e^{A^*(T-s)}\xi|^2\,ds \tag{2.2.5}$$

where $(\cdot,\cdot)$, resp. $|\cdot|$ stand resp. for the Euclidean scalar product, resp. norm. Therefore, the matrix P is invertible iff $B^* e^{A^*(T-s)}\xi \not\equiv 0$ for any $\xi \neq 0$, where s runs over $[0, T]$. However, $B^* e^{A^* t}\xi \equiv 0$ is equivalent to $(e^{At}By, \xi) \equiv 0$ for any $y \in U = \mathbf{R}^m$. It is clear that the linear span

$$W = \sum_i e^{At_i}By_i \tag{2.2.6}$$

of vectors of the form $e^{At}By$, where $y \in U$, contains BU and is A-invariant. Therefore, it coincides with the phase space V by virtue of controllability. On the other hand, W is orthogonal to ξ. We have to conclude, that $\xi = 0$, and P is invertible. ▲

Exercise. Prove the following generalization of Theorem 2.2.1

Theorem 2.2.2 *One can get from the origin to any given point $x$of the attainable set $\mathcal{D}$ at any given time $T > 0$ by using a control of the form (2.2.3).*

Corollary 2.2.3 *Consider any positive time instant $T > 0$, initial set M and attainable sets $\mathcal{D}(T)(M)$, and $\mathcal{D} = \bigcup_{t>0} \mathcal{D}(t)(M)$ for the system (7.3.2). Then $\mathcal{D} = \mathcal{D}(T)(M)$.*

Prof is left to the listener as an exercise. ▲

Chapter 3

The Kalman criterion III

In this chapter our object are Linear systems with bounded controls.

We consider first the control systems of the form

$$\dot{x} = Ax + Bu, \quad u \in U, \ |u| \leq 1 \tag{3.0.1}$$

So, the only difference with the unconstrained Kalman systems is the condition $|u| \leq 1$. One can deal in a very similar fashion with the constraint of the form $u \in \mathcal{U} \subset U$, where $\mathcal{U}$ is a convex body in U.

We study the systems (4.1.1) for two reasons: because they are important and interesting by themselves, and because they help study the unconstrained Kalman systems.

3.1 Attainable sets for constrained and unconstrained systems

Denote by $\mathcal{D}$ the attainable set $\mathcal{D}(T)(\{0\})$ of the unconstrained Kalman system

$$\dot{x} = Ax + Bu, \quad u \in U. \tag{3.1.1}$$

Note that by the results of the previous lecture, this set does not depend on $T > 0$. By $\mathcal{D}_c(T)$ we denote the attainable set $\mathcal{D}(T)(\{0\})$ of the constrained system (4.1.1). Here, the subscript c stands for constrained, and the dependence on time T is very real.

The basic relation between $\mathcal{D}$ and $\mathcal{D}(T)(\{0\})$ is this:

Theorem 3.1.1 *The set $\mathcal{D}_c(T)$ is a compact convex body in the vector space $\mathcal{D}$.*

First, we need to explain the meaning of the terms used. We use the term compactness as equivalent of sequential compactness. Thus, the compactness in Theorem 3.1.1 means that if $x_n \in \mathcal{D}_c(T)$ is a sequence of points, then there exists a subsequence (which we denote, by abusing the language slightly, by) $x_m \in \mathcal{D}_c(T)$, which converges $x_m \to x \in \mathcal{D}_c(T)$. Convexity of a set Ω means that Ω contains the entire interval $[x', x''] = \{x = \lambda x' + (1-\lambda)x'',\ \lambda \in [0,1]\}$ as soon as both $x', x'' \in \Omega$. Finally, a convex Ω is said to be a body, if it has a non-void interior.

We start proving Theorem 3.1.1 by observing that

1. $\bigcup_{\epsilon>0} \frac{1}{\epsilon}\mathcal{D}_c(T) = \mathcal{D}$

In other words, if $x \in \mathcal{D}$, then there exists $\epsilon > 0$ such that $\epsilon x \in \mathcal{D}_c(T)$. This is true because one can reach x from 0 by using a bounded control $|u| \le M$ (see the previous lecture). Now it remains to put $\epsilon = 1/M$, and reach ϵx by using the control $|\epsilon u| \le 1$.

The convexity of $\mathcal{D}_c(T)$ is easy: Suppose, $x_i \in \mathcal{D}_c(T)$, $i = 1, 2$ and $x_i = x^{u_i}(T)$, where $|u_i| \le 1$. Then, $\frac{x_1+x_2}{2} = x^u(T)$, where $u = \frac{u_1+u_2}{2}$.

The compactness statement requires some functional analysis to prove. More precisely, we need the following (slight restatement of) Theorem of Alaoglu-Bourbaki

Theorem 3.1.2 *Suppose $X = Y^*$ is a Banach space, which is dual (is the space of continuous linear functionals) of another Banach space. Let $B \subset X$ is a ball in X, and $b_n \in B$ is a sequence of points of B. Then, there exists a subsequence $\{b_m\}$ and point $b \in B$ such that $b(y) = \lim_{m\to\infty} b_m(y)$ for any $y \in Y$. (Here, $b(y)$ means the value of the functional b in y.)*

We use the Alaoglu-Bourbaki theorem in the following setup: $X = L_\infty(0,T)$ is the set of bounded controls (vector-functions with value in U), $Y = L_1(0,T)$ is the set of integrable vector functions, and B is the set of admissible controls $|u| \le 1$.

Now, suppose we are given a sequence $x_n \in \mathcal{D}_c(T)$, and $u_n \in B$ is the corresponding sequence of admissible controls. By Alaoglu-Bourbaki there exists an admissible control $u \in B$ such that

$$\int_0^T (u(s), v(s))\, ds = \lim_{m\to\infty} \int_0^T (u_m(s), v(s))\, ds \tag{3.1.2}$$

for any fixed integrable function v. This implies, that

$$\int_0^T e^{A(T-s)}Bu(s)\,ds = \lim_{m\to\infty}\int_0^T e^{A(T-s)}Bu_m(s)\,ds \qquad (3.1.3)$$

because each component of the matrix function $s \mapsto e^{A(T-s)}B$ is obviously integrable. It remains to note, that the equality (3.1.3) means that the subsequence $x_m \in \mathcal{D}_c(T)$ converges to $x = x^u(T) \in \mathcal{D}_c(T)$.

One might abstain from using a relatively abstract tool like the Alaoglu-Bourbaki theorem by invoking instead more direct arguments: Consider a sequence of admissible controls $u_n(t)$ such that the ends $x_n(T)$ of corresponding trajectories converge to point $p \in \mathbf{R}^n$. We have to show that $p = x(T)$, where $x(t)$ is an admissible trajectory. To find x we choose a subsequence of controls u_n such that the corresponding trajectories x_n converge uniformly on $[0,T]$. This is possible by the Ascoli-Arcela theorem, because the functions x_n are uniformly Lipschitz. Denote by x the limit of x_n over the subsequence, and prove that it is an admissible curve. We have for any $h > 0$

$$\frac{1}{h}(x_n(t+h) - x_n(t)) = \frac{1}{h}\int_t^{t+h} Ax_n(s)ds + \frac{1}{h}\int_t^{t+h} Bu_n(s)ds$$

Note, that all terms in the formula, save for the last one, converge as $n \to \infty$. The last term has the form Bv_n, where $|v_n| \le 1$. Indeed, $v_n = \frac{1}{h}\int_t^{t+h} u_n(s)ds$, and $|u_n(s)| \le 1$. This implies, that

$$\frac{1}{h}(x(t+h) - x(t)) = \frac{1}{h}\int_t^{t+h} Ax(s)ds + Bv,$$

where $v = v(t,h)$ satisfies the bound $|v| \le 1$. Therefore, by passing to the limit $h \to 0$, we obtain that at each point, where $x(t)$ is differentiable, we have $\dot{x} = Ax + Bv$, where $|v| \le 1$. Thus, $x(t)$ is an admissible curve, and $p = x(T)$.

Now, to prove Theorem 3.1.1 it remains to deduce from the statement **1** that $\mathcal{D}_c(T)$ has a non-void interior in $\mathcal{D}$. To do so, we choose a basis e_i of the vector space $\mathcal{D}$, and $\epsilon > 0$ such that $\pm\epsilon e_i \in \mathcal{D}_c(T)$ for all i. It is clear, that the convex hull of the set $\{\pm\epsilon e_i\}$ is a hypercube in $\mathcal{D}$ with nonempty interior. ▲

Corollary 3.1.3 *The system (7.3.1) is completely controllable iff the attainable sets $\mathcal{D}_c(T)$ of the constrained system (4.1.1) are convex* bodies.

Exercise. Prove Theorem 3.1.1 when the control bound has the form $u \in \mathcal{U} \subset U$, where $\mathcal{U}$ is a convex body in U.

3.2 Convex sets and support functions

An important tool for study the convex sets is provided by the support function.

Definition 3.2.1 *Let Ω be a subset of finite-dimensional vector space V with the dual space V^*. The support function H_Ω is defined by*

$$H_\Omega(\xi) = \sup_{x\in\Omega}(x,\xi) \tag{3.2.1}$$

Here, $\xi \in V^*$, (x,ξ) is the value of the functional ξ in x. If $V = \mathbf{R}^n$ we identify V and V^*, and regard H_Ω as a function on V.

The geometric meaning of H_Ω is this: the support hyperplane

$$\{x \in V; (x,\xi) = H_\Omega(\xi)\}$$

is a kind of "tangent" hyperplane to the boundary of Ω with the outer normal vector ξ. So, $H_\Omega(\xi)$ measures distance from the origin to the "tangent" hyperplane if $|\xi| = 1$. Here, we put the word tangent into the quotation marks, because Ω might not have a smooth boundary.

One can see easily that H_Ω in fact depends not on Ω, but rather on the closure of its convex hull $\overline{\operatorname{conv}}\,\Omega$.

Indeed, it is clear that closing a set does not affect its support function. As to the passing to the convex hull $\Omega \mapsto \operatorname{conv}\Omega$, one obviously has $H_\Omega \le H_{\operatorname{conv}\Omega}$ since $\Omega \subset \operatorname{conv}\Omega$. The reverse inequality means that for any $x \in \operatorname{conv}\Omega$ and ξ there exists $y \in \Omega$ such that $(y,\xi) \ge (x,\xi)$. However, any $x \in \operatorname{conv}\Omega$ can be represented in the form

$$x = \lambda y_1 + (1-\lambda)y_2, \tag{3.2.2}$$

where $y_i \in \Omega$, and $\lambda \in [0,1]$. One can see immediately, that both inequalities $(y_i,\xi) < (x,\xi)$, $i = 1,2$ are incompatible with (3.2.2). This proves the reverse inequality $H_\Omega \ge H_{\operatorname{conv}\Omega}$, and, thus, the equality $H_\Omega = H_{\operatorname{conv}\Omega}$.

In fact, the support functions characterize closed convex sets completely, but we postpone the discussion till the next lecture.

3.3 The support function of attainable set

Here, we compute explicitly the support function $H_{\mathcal{D}_c(T)}$. By the Cauchy formula

$$H_{\mathcal{D}_c(T)}(\xi) = \sup_{|u|\le 1} \int_0^T (e^{A(T-s)}Bu(s), \xi)\, ds \tag{3.3.1}$$

One can put the sup sign under the integral sign, because the bounds $|u(s)| \le 1$ are "independent" for different s. Thus,

$$H_{\mathcal{D}_c(T)}(\xi) = \int_0^T \sup_{|u(s)|\le 1} (e^{A(T-s)}Bu(s), \xi)\, ds \tag{3.3.2}$$

It is clear that

$$\sup_{|u|\le 1} (e^{A(T-s)}Bu, \xi) = \sup_{|u|\le 1} (u, B^* e^{A^*(T-s)}\xi) = |B^* e^{A^*(T-s)}\xi|$$

Note that the last equality can be regarded as a computation of the support function of a unit ball. Therefore,

$$H_{\mathcal{D}_c(T)}(\xi) = \int_0^T |B^* e^{A^*(T-s)}\xi|\, ds = \int_0^T |B^* e^{A^* s}\xi|\, ds \tag{3.3.3}$$

and our goal is achieved.

Exercise. State and prove an explicit formula for the support function of attainable set when the control bound has the form $u \in \mathcal{U} \subset U$, where $\mathcal{U}$ is a convex body in U, and initial set M is arbitrary.

Chapter 4

The Kalman criterion IV

This chapter is faced to additional considerations ont convex sets. We speak about convex sets, supported functions of attainable sets of linear systems, and duality between controllability and observability.

4.1 More about convex sets

Remind that in the previous lecture we considered the control systems of the form

$$\dot{x} = Ax + Bu, \quad u \in U, \ |u| \leq 1 \tag{4.1.1}$$

The only difference with the unconstrained Kalman systems

$$\dot{x} = Ax + Bu, \quad u \in U. \tag{4.1.2}$$

is the condition $|u| \leq 1$. One can deal in a very similar fashion with the constraint of the form $u \in \mathcal{U} \subset U$, where $\mathcal{U}$ is a convex body in U.

Corollary 4.1.1 *The attainable sets $\mathcal{D}_c(T)$ of the constrained system (4.1.1) are convex* bodies *in the attainable sets $\mathcal{D}$ of unconstrained system (7.3.1).*

Corollary 4.1.2 *The system (7.3.1) is completely controllable iff the attainable sets $\mathcal{D}_c(T)$ of the constrained system (4.1.1) are convex* bodies.

4.2 More on support functions

Main properties of the support function $H_\Omega(\xi)$ are:

1. The support function is homogeneous of degree 1: $H_\Omega(\lambda\xi) = \lambda H_\Omega(\lambda\xi)$, if $\lambda > 0$

2. The support function is convex:

$$H_\Omega(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda H_\Omega(\xi_1) + (1-\lambda)H_\Omega(\xi_2),$$

if $\lambda \in [0,1]$

Exercise. Prove this.

Theorem 4.2.1 *The correspondence $\Omega \mapsto H_\Omega$ establishes a bijection between closed convex sets and functions with values in $\mathbf{R} \cup +\infty$ with properties* **1, 2**. *It also establishes a bijection between compact convex sets and real functions with properties* **1, 2**.

Proof. We have to restore Ω from H_Ω, and we do this by the following criterion:

$$x \in \Omega \Longleftrightarrow (x,\xi) \leq H_\Omega(\xi), \ \forall \xi \tag{4.2.1}$$

The proof of $\Longrightarrow$ is obvious. In the opposite direction we have to show that if $x \notin \Omega$, then there exists $\xi \neq 0$ such that $(x,\xi) > H_\Omega(\xi)$. To get such a ξ, we first find a closest to x point $y \in \Omega$. There is a unique such point. Indeed, to prove existence we take any point $z \in \Omega$, and put $a = |x-z|$, $K = \Omega \bigcap \{v \in V; |x-v| \leq a\}$. Then K is a (convex) compact, and the function $z \mapsto |x-z|$ is continuous and strictly convex. Its minimum is attained at a point $y \in K$, and at the same time y is a closest to x point $y \in \Omega$ (easy exercise). If there are two such points y_i, $i = 1,2$, then the point $y = (y_1+y_2)/2 \in \Omega$ is closer to x, then y is closer to x than both y_i. Thus, the closest point y exists and is unique.

Now, we put $\xi = x - y$. If $z \in \Omega$ is an arbitrary point, then $(z-y,\xi) \leq 0$. Indeed, otherwise we would get $|y + \epsilon(z-y) - x|^2 < |y-x|^2$ for sufficiently small ϵ, which, in view of $y + \epsilon(z-y) \in \Omega$, contradicts the fact that y is closest to x.

Now, $(z,\xi) \leq (y,\xi) = (x,\xi) - (\xi,\xi) < (x,\xi)$ which proves that $(x,\xi) > H_\Omega(\xi)$. ▲

Note also, that the computation of $H_{\mathcal{D}_c(T)}(\xi)$ is equivalent to the following control problem:
$(x(T),\xi) \to \sup$, where $x(t)$ is an admissible curve for (**??**) such that $x(0) = 0$
Exercise. Solve the problem by using the maximum principle.

One can give a useful formula for the boundary of a convex set with a smooth support function.

Theorem 4.2.2 *Suppose, $H_\Omega(\xi)$ is C^1-smooth for $\xi \neq 0$, and $x(\xi) \in \partial\Omega$ is a support point for ξ: i.e., $(x(\xi), \xi) = H_\Omega(\xi)$. Then the point $x(\xi)$ is unique, and is given by $x(\xi) = \frac{\partial H_\Omega(\xi)}{\partial \xi}$.*

Proof. Indeed, if we put $x = \frac{\partial H(\xi)}{\partial \xi}$, then $(x, \xi) = H(\xi)$ by the Euler theorem ($H = H_\Omega$ is homogeneous), and $(x, \eta - \xi) \leq H(\eta) - H(\xi)$ for any η, because H is convex. This implies that $(x, \eta) \leq H(\eta)$ and, therefore, $x \in \Omega$, and x is a support point for ξ.

Conversely, if $x \in \Omega$, and $(x, \xi) = H(\xi)$, then $(x, \eta - \xi) \leq H(\eta) - H(\xi)$ for any η. If H is differentiable this implies $x = \frac{\partial H(\xi)}{\partial \xi}$. ▲

4.3 Kalman duality

We consider, following Kalman, the issue of observability of a linear system. In general, the problem is to restore the unknown curve $t \mapsto x(t)$, usually the phase curve of a differential equation, by observing the curve $t \mapsto h(x(t))$, where h is a known given map. In the time-invariant linear setting, the problem is:

$$\begin{cases} \dot{x} = Cx \\ y = Dx \end{cases} \tag{4.3.1}$$

y is the observable curve, C and D are known matrices, and we wish to restore x from y. The system is said to be observable if this restoration is always possible. On the other hand, if the system (4.3.1) is not observable, this implies the existence of *undetectable states* x such that $De^{Ct}x \equiv 0$. Indeed, the output is $\equiv 0$ and restoration of x is impossible.

Kalman have found a relation between controllability and observability of "dual" systems: the system (**??**) and the system

$$\begin{cases} \dot{p} = -A^* p \\ y = B^* p \end{cases} \tag{4.3.2}$$

(Notice the connection with the maximum principle)

Kalman theorem is as follows:

Theorem 4.3.1 *Let $\mathcal{P}$ is the set of undetectable states for (6.1.13), and $\mathcal{D}$ is the attainable set (from the origin) for (**??**). Then, $\mathcal{P}$ and $\mathcal{D}$ are the orthogonal complements of each other: $\mathcal{P}^\perp = \mathcal{D}$, and $\mathcal{D}^\perp = \mathcal{P}$.*

Corollary 4.3.2 *The system (??) is controllable iff the system (6.1.13) is observable.*

Corollary 4.3.3 *The system (4.3.1) is observable iff the rank of the matrix*

$$K = \begin{bmatrix} D & DC & \dots & DC^{n-1} \end{bmatrix}^* \tag{4.3.3}$$

is maximal possible.

We give here a somewhat exotic proof, based on the above study of linear control systems with bounded control. It follows from the formula for the support function of $\mathcal{D}_c(T)$ that $H_{\mathcal{D}_c(T)}(\xi) = 0$ iff ξ is undetectable vector for (6.1.13). Indeed, $H_{\mathcal{D}_c(T)}(\xi) = \int_0^T |B^* e^{A^* t}\xi|\, dt$, and $\int_0^T |B^* e^{A^* t}\xi|\, dt = 0$ is equivalent to

$$B^* e^{A^* t}\xi \equiv 0 \text{ for } t \in [0, T].$$

By the principle of the analytic continuation this is equivalent to $B^* e^{A^* t}\xi = 0$ for any t

On the other hand, it follows from the Corollary 4.1.1 that $H_{\mathcal{D}_c(T)}(\xi) = 0$ iff $\xi \perp \mathcal{D}$. Indeed, if $\xi = \xi' + \xi''$ is the decomposition of ξ, where $\xi' \in \mathcal{D}$, and ξ'' is orthogonal to $\mathcal{D}$, then $H_{\mathcal{D}_c(T)}(\xi) = H_{\mathcal{D}_c(T)}(\xi')$, which is $\geq c|\xi'|$ since $\mathcal{D}_c(T)$ contains a ball centered in the origin. ▲

Exercise. Prove the Kalman duality by pure linear algebra, without recourse to convex sets, and auxiliary control problems. (Hint: the space $\mathcal{P}$ of undetectable vectors for (6.1.13) can be characterized as the maximal A^*-invariant subspace of $\ker B^*$, while the attainable set $\mathcal{D}$ is the minimal A-invariant subspace of the entire phase space containing $\operatorname{Im} B$. Since $\ker B^* = \operatorname{Im} B^\perp$ we conclude that $\mathcal{P} = \mathcal{D}^\perp$.)

Chapter 5

Brammer's criterion

Here, we turn to controllability of linear systems with bounded control

$$\dot{x} = Ax + Bu, \quad u \in U, \ |u| \leq 1 \tag{5.1.1}$$

The only difference with the unconstrained Kalman systems

$$\dot{x} = Ax + Bu, \quad u \in U. \tag{5.1.2}$$

is the condition $|u| \leq 1$. One can deal in a very similar fashion with the constraint of the form $u \in \mathcal{U} \subset U$, where $\mathcal{U}$ is a convex body in U such that 0 is contained in the interior of $\mathcal{U}$.

Theorem 5.1.4 *The system (5.1.1) is completely controllable iff*

1. *The Kalman criterion holds (i.e., the system (5.1.2) is completely controllable)*
2. *The spectrum of the matrix A is imaginary*

Proof. We start with the necessity of conditions 1,2. The necessity of 1 is obvious. To prove necessity of 2 we assume first that there exists an eigenvalue λ of A such that $\operatorname{Re}\lambda < 0$, and show that this is incompatible with controllability of (5.1.1).

To do this we take a closer look at the formula for the support function of the attainable set $D(T) = \mathcal{D}(T)(\{0\})$ for (5.1.1)

$$H_{D(T)}(\xi) = \int_0^T \left| B^* e^{A^* t} \xi \right| \, dt \tag{5.1.3}$$

If η is a λ-eigenvector of A^* then, the complex conjugate vector $\overline{\eta}$ is a $\overline{\lambda}$-eigenvector, and because of $\operatorname{Re}\lambda < 0$ the integrand in (5.1.3) is exponentially decreasing for any ξ of the form $a\eta + \overline{a\eta}$, where $a \in \mathbf{C}$. It is almost obvious that $a\eta + \overline{a\eta} \neq 0$ for some a (provide details!). Therefore, $H_{D(T)}(\xi)$ remains bounded as $T \to \infty$ for such $\xi \neq 0$. This is incompatible with controllability.

Since controllability is invariant under the time reversal, we conclude that $\operatorname{Re}\lambda > 0$ also is incompatible with controllability.

Thus, the necessity of 2 is proved.

Now we assume 1,2, and prove that (5.1.1) is completely controllable.

We already know that

$$H_{D(T)}(\xi) > 0 \text{ as soon as } \xi \neq 0$$

because this is one of the reincarnations of the Kalman criterion. What we have to establish is that

$$H_{D(T)}(\xi) \to +\infty, \tag{5.1.4}$$

for any $\xi \neq 0$ as $T \to \infty$. Indeed, suppose x is any fixed state vector; then $x \in D(T)$ iff $(x,\xi) \leq H_{D(T)}(\xi)$ for all ξ. Denote by $M_T = M_T(x)$ the compact set

$$M_T(x) = \{\xi : |\xi| = 1,\ (x,\xi) \geq H_{D(T)}(\xi)\} \tag{5.1.5}$$

The intersection $\bigcap_{T>0} M_T$ is empty because of (5.1.4), which means that $M_T = M_T(x)$ is empty for sufficiently large T. Therefore, for sufficiently large T we have

$$(x,\xi) \leq H_{D(T)}(\xi),\ \forall \xi \tag{5.1.6}$$

which means that $x \in D(T)$.

Therefore, (5.1.4) implies that one can reach any point from the origin. Since our basic conditions 1,2 are invariant under time reversal, we are done.

To prove (5.1.4) we should look more closely at the explicit expression (5.1.3), especially at the integrand

$$f(t) = B^* e^{A^* t}\xi = \sum_{j,\omega} t^j a_{j,\omega} e^{i\omega t}, \tag{5.1.7}$$

where the sum is finite, all ω are real, and $a_{j,\omega}$ is a vector. We can rewrite (5.1.7) in the form

$$f(t) = \sum_j t^j \sum_\omega a_{j,\omega} e^{i\omega t} = \sum_j t^j f_j(t), \tag{5.1.8}$$

where each $f_j(t) = \sum_\omega a_{j,\omega} e^{i\omega t}$ is a (non-zero) trigonometric polynomial.

We will show that

$$cT^{j+1} \le \int_0^T t^j |f_j(t)|\, dt \le CT^{j+1}, \tag{5.1.9}$$

where c, C are positive constants. Note, that the right-hand side inequality here is trivial. This would imply (in view of the triangle inequality for L_1-norm) that

$$cT^{N+1} \le \int_0^T |f(t)|\, dt \le CT^{N+1}, \tag{5.1.10}$$

where N is the maximum of j. In particular, this would prove (5.1.4).

Thus, everything is reduced to the inequality

$$cT^{j+1} \le \int_0^T t^j |\phi(t)|\, dt \le CT^{j+1}, \tag{5.1.11}$$

where $\phi(t) = \sum_\omega a_\omega e^{i\omega t} \ne 0$ is a (vector-valued) trigonometric polynomial.

Note, that only left inequality (5.1.11) is nontrivial; the right one follows immediately from the trivial bound $|\phi(t)| \le C$.

To prove the left inequality (5.1.11), we use the trivial estimate:

$$\int_0^T t^j |\phi(t)|^2\, dt \le C \int_0^T t^j |\phi(t)|\, dt \tag{5.1.12}$$

and our task is reduced to proving

$$\int_0^T t^j |\phi(t)|^2\, dt \sim T^{j+1}. \tag{5.1.13}$$

The basic advantage of (5.1.13) compared to (5.1.11) is that one can compute the integral in the former equation explicitly, which is hardly possible in the latter one. Indeed,

$$\int_0^T t^j |\phi(t)|^2\, dt = \int_0^T t^j \left(\sum_\omega |a_\omega|^2 + \sum_{\omega' \ne \omega''} (a_{\omega'}, a_{\omega''}) e^{i(\omega' - \omega'')t} \right) dt \tag{5.1.14}$$

An easy computation (exercise in integration by parts) shows that

$$\int_0^T t^j e^{i(\omega' - \omega'')t}\, dt = O(T^j) \tag{5.1.15}$$

Therefore, the only essential contribution (as $T \to \infty$) to the integral (6.1.10) is given by

$$\sum_{\omega} |a_\omega|^2 \int_0^T t^j \, dt = CT^{j+1} \tag{5.1.16}$$

which proves (5.1.13). ▲

We have chosen a most elementary method of proving the crucial inequality (5.1.4). In fact, a better control over the asymptotic behavior of $H_{D(T)}$ can be achieved by using averaging in the spirit of [1].

Chapter 6

Stochastic approach

6.1 The Least square method

Our exposition follows primarily the paper [23].

6.1.1 The problem statement

The simplest setup for the least square method (LSM) is this:

$$y = \sum_{j=1}^{n} a_j x_j \tag{6.1.1}$$

Here x_j are known reals, a_j are unknown coefficients of a linear form, y is observable with an additive error Δ. In other words, we know exactly $\eta = y + \Delta$. The problem is to find a good estimator of the vector $a = (a_j)$, $j = 1, \dots, n$.

More precisely, the relation (7.1.3) describes an "experiment", and we are conducting many, say, N experiments

$$y_r = \textstyle\sum_{j=1}^{n} a_j x_{jr},\ \eta_r = y_r + \Delta_r,\ r = 1, \dots, N \tag{6.1.2}$$

We define vectors $y, \Delta, x_i \in \mathbf{R}^N$ by $y = (y_r)$, $\Delta = (\Delta_r)$, $x_i = (x_{ir})$, and the matrix $\mathcal{X} = (x_{ir})$. Here, $i = 1, \dots, n$, $r = 1, \dots, N$. We will use in this lecture a nonstandard notation

$$[xy] = \sum_{i=1}^{N} x_i y_i \tag{6.1.3}$$

due to Gauss for the scalar product of vectors $x, y \in \mathbf{R}^N$. This way we are paying tribute to the inventor of LSM.

We assume that the following equivalent conditions are met:

$$\begin{array}{ll} \text{(i)} & \mathrm{rk}\mathcal{X} = n \\ \text{(ii)} & \text{vectors } x_i \text{ are linearly independent} \\ \text{(iii)} & \det([x_i x_j]) \neq 0 \end{array} \tag{6.1.4}$$

Exercise 6.1.1 *Prove the equivalence of the conditions (6.5.3).*

6.1.2 The algorithm

We have to find an estimator $\alpha = (\alpha_j) = (\alpha_j(\eta))$ for the vector $a = (a_j)$, $j = 1, \ldots, n$. To do so we introduce, following Gauss, the vector $\varepsilon = \varepsilon(\alpha)$ by

$$\varepsilon_r = \eta_r - \sum_{j=1}^{n} \alpha_j x_{jr} \tag{6.1.5}$$

Then we find α as a solution to the following optimization problem:

$$[\varepsilon(\alpha)\varepsilon(\alpha)] \to \min_{\alpha \in \mathbf{R}^n}, \tag{6.1.6}$$

where the minimization is performed over all $\alpha \in \mathbf{R}^n$. The condition (6.1.6) is equivalent to the following:

$$\sum_{j=1}^{n} [x_i x_j] \alpha_j = [x_i \eta],\ i = 1, \ldots, n \tag{6.1.7}$$

Geometrically, (6.1.7) says that the vector ε is orthogonal to all x_i (prove this!).

Exercise 6.1.2 *Prove the equivalence of the conditions (6.1.6) and (6.1.7).*

Exercise 6.1.3 *Prove that the vector α is defined uniquely by (6.1.6) or (6.1.7).*

6.1.3 Properties of LSM

Assume that the observation error Δ_r, $\forall r$ is a random variable with a zero mean

$$\mathbf{E}\Delta_r = 0, \tag{6.1.8}$$

then

$$\mathbf{E}\alpha = a \tag{6.1.9}$$

In other words, α_j is an unbiased estimator of a_j.

Indeed, it follows from (6.1.7) and (6.1.2) that

$$\sum_j [x_i x_j](\alpha_j - a_j) = [x_i \Delta] \tag{6.1.10}$$

and, therefore, $\sum_j [x_i x_j](\mathbf{E}\alpha_j - a_j) = 0$ which implies $\mathbf{E}\alpha_j - a_j = 0$.

Now we make a next assumption about the structure of observation errors:

$$\mathbf{E}\Delta_i\Delta_j = \delta_{ij}s^2,\ i = 1, \ldots, N \tag{6.1.11}$$

Then

$$\mathbf{E}(\alpha_i - a_i)(\alpha_j - a_j) = q_{ij}s^2,\ i = 1, \ldots, N, \tag{6.1.12}$$

where the matrix $q = G^{-1}$, where $G_{ij} = [x_i x_j]$ is the Gram matrix of the vectors x_i.

To prove (6.1.12) we develop some (limited) linear-algebraic machinery. Let $L = [x_1, \ldots, x_n]$ be the vector space generated by vectors x_i. In view of (6.5.3), the set $\{x_i\}_{i=1}^n$ is a basis of L. The dual basis $\{u_j\}_{j=1}^n$ is defined via

$$[x_i u_j] = \delta_{ij}. \tag{6.1.13}$$

Next, we can interpret (6.1.10) as follows:

$$\sum_j x_j(\alpha_j - a_j) = P_L\Delta, \tag{6.1.14}$$

where P_L is the orthogonal projection on L. This immediately implies

$$\alpha_i - a_i = [(P_L\Delta)u_i] = [\Delta u_i]. \tag{6.1.15}$$

Now, we have

$$\mathbf{E}(\alpha_i - a_i)(\alpha_j - a_j) = \mathbf{E}[\Delta u_i][\Delta u_j] = s^2 \sum_{s=1}^n u_{is}u_{js} = s^2[u_i u_j], \tag{6.1.16}$$

(explain the second equality in (6.1.16)!) and it remains to show that

$$[u_i u_j] = q_{ij}. \tag{6.1.17}$$

In fact one can see immediately that

$$x_i = \sum [x_i x_j] u_j, \tag{6.1.18}$$

which implies that

$$u_i = \sum q_{ij} x_j. \tag{6.1.19}$$

Now, (6.1.17) follows from (6.1.19).

We can say more about the size of the vector ε, namely,

$$\mathbf{E}[\varepsilon\varepsilon] = (N-n)s^2, \tag{6.1.20}$$

which follows from $\epsilon = \Delta - P_L \Delta$

Exercise 6.1.4 *Prove (6.1.20).*

One can interpret (6.1.20) as follows: If the covariance s^2 is unknown, its *unbiased* estimator is given by σ^2, where

$$\sigma = \sqrt{[\varepsilon\varepsilon]/(N-n)} \tag{6.1.21}$$

6.1.4 Gaussian case

If the error vector Δ is Gaussian (Gaussian case) one can say more about properties of LSM. For instance,

$$D\sigma^2 := \mathbf{E}(\sigma^2 - s^2)^2 = 2s^4/(N-n) \tag{6.1.22}$$

This can be reduced to the following statement: Put $\chi^2_m = \sum_{i=1}^m \xi_i^2$, where ξ_i are independent Gaussian variables with zero mean and unit covariance:

$$\xi_i \in \mathcal{N}(0,1). \tag{6.1.23}$$

Then,

$$D\chi^2_m = \mathbf{E}(\chi^2_m - m)^2 = 2m. \tag{6.1.24}$$

Exercise 6.1.5 *Make a reduction of the statement (6.1.22) to (6.1.24), where* $m = N - n$.

To prove (6.1.24) we calculate

$$\mathbf{E}(\chi_m^2 - m)^2 = \sum_{i=1}^{m} \mathbf{E}(\xi_i^4) + 2\sum_{i<j} \mathbf{E}(\xi_i^2)\mathbf{E}(\xi_j^2) - m^2. \tag{6.1.25}$$

We have $\mathbf{E}(\xi_k^2) = 1$, since ξ_k are independent Gaussian variables with zero mean and unit covariance. Furthermore,

$$\mathbf{E}(\xi_k^4) = \left(\frac{\partial}{\partial t}\right)^4 \Bigg|_{t=0} \mathbf{E}e^{it\xi_k} = \left(\frac{\partial}{\partial t}\right)^4 \Bigg|_{t=0} e^{-\frac{1}{2}t^2} = \frac{4!}{8} = 3.$$

By using the above identities in (6.1.25) we obtain

$$\mathbf{E}(\chi_m^2 - m)^2 = 3m + 2\frac{m(m-1)}{2} - m^2 = 2m. \tag{6.1.26}$$

6.1.5 The Student distribution

In fact, in the Gaussian case we have

$$\frac{\alpha_j - a_j}{\sqrt{q_{jj}}s} \in \mathcal{N}(0,1). \tag{6.1.27}$$

Exercise 6.1.6 *Prove (6.1.27).*

In case the deviation s is unknown, one can use $\phi = \frac{\alpha_j - a_j}{\sqrt{q_{jj}}\sigma}$, where σ is defined in (6.1.21), instead of $\frac{\alpha_j - a_j}{\sqrt{q_{jj}}s}$. Then the distributional density of ϕ takes the form

$$p_\phi(x) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\Gamma(\frac{m}{2})}\left(1 + \frac{x^2}{m}\right)^{-\frac{m+1}{2}}, \tag{6.1.28}$$

where $m = N - n$.

Exercise 6.1.7 *Prove (6.1.28).*

The r.h.s. of (6.1.28) is called the Student distribution.

Exercise 6.1.8 *Check that the Student distribution tends (weakly) to the standard Gaussian distribution $e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ as $m \to \infty$.*

One can check that it tends (weakly) to the standard Gaussian distribution $e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ as $m \to \infty$.

6.1.6 Maximum likelihood and least square

Maximum likelihood estimator might be defined in many situations. In the case of observations (6.1.2) it is defined as follows: Consider the value $p_\alpha(\eta)$ at $x = \eta$ of the density $p_\alpha(x)$ of the random variable $\eta_\alpha := \sum_j \alpha_j x_j + \Delta$. Here, η is the result of observation, and α is unknown parameter. The maximum likelihood estimator (MLE) solves the following optimization problem:

$$p_\alpha(\eta) \to \min_{\alpha \in \mathbf{R}^n} . \tag{6.1.29}$$

Exercise 6.1.9 *Suppose that the error vector Δ is Gaussian. Check that the MLE coincides with LSM.*

Conditional Expectation and Least Square. Another related general concept is that of conditional expectation. The problem is like this: Suppose, that the unknown vector a is somehow stochastic, so that one can speak of the joint distribution of $\{\eta, a\}$ and of the conditional expectation $\mathbf{E}(a|\eta)$ of a, provided that η is known. What kind of the probabilistic setup should it be, in order to guarantee that $\mathbf{E}(a|\eta)$ can be found via LSM?

One possible answer is that a should be uniformly distributed in $\mathbf{R}^n$ and independent of Δ. In order to state a precise result we approximate the uniform distribution by the Gaussian one with a very large covariance:

$$p_a(x) = \epsilon^{n/2}(2\pi)^{-n/2} e^{-\frac{1}{2}\epsilon|x|^2},\ x \in \mathbf{R}^n \tag{6.1.30}$$

Exercise 6.1.10 *Explain why the distribution $p_a(x)$ approximates the uniform one.*

Exercise 6.1.11 *Suppose that the error vector Δ is Gaussian and independent of a, where a is distributed according to (6.1.30). Check that $\lim_{\epsilon\to\infty} \mathbf{E}(a|\eta)$ coincides with the LSM estimator α.*

6.2 Stochastic Integrals

6.2.1 Definition of the integral

The problem we will deal with is to define the value of

$$\int f(s)\xi(s)ds \stackrel{\text{def}}{=} \int f(s)dw(s), \tag{6.2.1}$$

where f is a suitable random function, ξ is the white noise, and w is the corresponding Wiener process. This is not obvious: By the very definition of the white noise the expression $\int f(s)\xi(s)ds$ is defined only if f is a deterministic function such that $\int |f(s)|^2 ds$ is finite. Moreover, the natural random function like $s \mapsto w(s)$ are not smooth, and its sample path $s \mapsto w(s,\omega)$ might be an "arbitrary" continuous function. Since it is known that the above sample path has an unlimited variation (the corresponding curve has no length), there is no way to define for each ω the value of $\int f(s,\omega)dw(s,\omega)$ by using the standard definition of the Lebesgue–Stiltjes integral. We have to somehow take into account an interaction between different sample paths.

It turns out that a natural class of random functions that allows for a good stochastic integration is that of non-anticipating functions.

We are given a white noise process ξ, and consider for definiteness only random functions defined on $[0,\infty)$. We define $\mathcal{F}_{\le t} = \sigma(\int_0^t f(s)\xi(s)ds)$ the σ-algebra generated by $\int_0^t f(s)\xi(s)ds$, where f is a smooth deterministic function. Similarly, we define $\mathcal{F}_{\ge t} = \sigma(\int_0^t f(s)\xi(s)ds)$ the σ-algebra generated by $\int_t^\infty f(s)\xi(s)ds$, where f is a smooth deterministic function. The non-anticipating functions are associated with a "flow" of σ-algebras $\mathcal{A}_t$, $t \in [0,\infty)$ (meaning that $\mathcal{A}_t \supset \mathcal{A}_s$ if $t \le s$) such that

- $\mathcal{A}_t \supset \mathcal{F}_{\le t}$,
- $\mathcal{A}_t$ is independent of $\mathcal{F}_{\ge t}$.

Heuristically, this means that the random variable $f(t)$ together with all $f(\tau)$, $\tau \le t$ and $\xi(\tau)$ is independent of all $\xi(s)$, $s \ge t$. Thus, the complete knowledge of f up to any time instant does not allow for any kind of prediction after the instant. It is worth mentioning also that the σ-algebras $\mathcal{A}_t$ are not canonical, and can be chosen properly for a particular problem, thus adding extra flexibility (which is usually not very important).

6.2.1.1. Integration of simple non-anticipating functions

A non-anticipating function f is called simple if there is such natural number n that $f(t) = f(2^{-n}[2^n t])$. Here $[x]$ stands for integer part of x, the maximal integer k such that $k \le x$. In other words, f is constant on the intervals of the form $[\frac{k}{2^n}, \frac{k+1}{2^n})$, where k is an integer.

Definition 6.2.1 *The stochastic integral* $I = \int_0^t f(s)\xi(s)ds = \int_0^t f(s)dw(s)$ *is defined as* $I = \sum_{k=0}^{l} f_k\Delta_k + f_l \int_{\frac{l}{2^n}}^t \xi(s)ds$, *where* $l = [2^n t]$, $f_k = f(\frac{k}{2^n})$ *and* $\Delta_k = \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \xi(s)ds$.

Exercise 6.2.2 *Check that the definition (6.2.1) is correct, meaning that it is independent of admissible* n. *If* n *is replaced by* $n+1$ *the value of* I *is the same.*

It is important that Δ_k in (6.2.1) is "directed towards future" with respect to the value $f(\frac{k}{2^n})$ of the integrand.

Proposition 6.2.3 *The basic properties of the "simple" stochastic integral are as follows:*

1. $\int_0^t (f+g)dw = \int_0^t fdw + \int_0^t gdw$,
2. $\int_0^t kfdw = k\int_0^t fdw$, *where* k *is a constant,*
3. *The integral depends continuously of* t, *and is non-anticipating,*
4. *The mathematical expectation* $\mathbf{E}\int_0^t fdw = 0$,
5. $\mathbf{E}(\int_0^t fdw)^2 = \mathbf{E}\int_0^t f(s)^2 ds$

Proof. The statements 1,2 are obvious. The statement 3 follows from the fact that the Wiener process is continuous. The proof of 4, 5 is easy but depends on non-anticipating property of the integrand in a crucial way. For simplicity, we assume in the proof that $t = 1$. The general case goes almost verbatim, but requires more complicated notations.

If $t = 1$, then $\int_0^t fdw = \sum_{k=0}^{2^n-1} f_k\Delta_k$, and to prove the statement 4 it suffices to show that $\mathbf{E}f_k\Delta_k = 0$. This is trivial, since due to non-anticipation $f(\frac{k}{2^n})$ is independent of Δ_k, and $\mathbf{E}\Delta_k = 0$.

To prove the statement 5 we write down

$$\left(\int_0^t fdw\right)^2 = \sum_{k=0}^{2^n-1} f_k^2\Delta_k^2 + 2\sum_{i<j} f_i f_j \Delta_i\Delta_j \tag{6.2.2}$$

Now it suffices to show that

$$\mathbf{E}\sum_{k=0}^{2^n-1} f_k^2\Delta_k^2 = \mathbf{E}\int_0^t f(s)^2 ds, \tag{6.2.3}$$

and that

$$\mathbf{E} f_i f_j \Delta_i \Delta_j = 0 \text{ if } i < j. \tag{6.2.4}$$

Identity (12.2.1) follows from $\mathbf{E}\Delta_k^2 = 2^{-n}$, and non-anticipation. Indeed, these arguments prove that

$$\mathbf{E} \sum_{k=0}^{2^n-1} f_k^2 \Delta_k^2 = \mathbf{E} \sum_{k=0}^{2^n-1} 2^{-n} f_k^2, \tag{6.2.5}$$

and it is obvious that

$$\sum_{k=0}^{2^n-1} 2^{-n} f_k^2 = \int_0^t f(s)^2 ds.$$

To prove (12.2.5) we have to notice that the random variables $\phi = f_i f_j \Delta_i$, resp. $\Delta = \Delta_j$ are independent, for they represent past, resp. future. Moreover, $\mathbf{E}\Delta = 0$, thus, $\mathbf{E}\phi\Delta = \mathbf{E}\phi\mathbf{E}\Delta = 0$.

6.2.1.2. Approximation by simple functions

Suppose that f is a non-anticipating function such that $\|f\|^2 := \mathbf{E}\int_0^t f(s)^2 ds$ is finite for any t. We will call this kind of functions tempered. Then f can be approximated by simple a non-anticipating functions f_n with respect to the norm $\|f\|$, meaning that $\|f - f_n\| \to 0$ as $n \to \infty$.

The standard construction of f_n is, in fact, purely deterministic, and goes via averaging over small intervals. For any (stochastic or deterministic) measurable function u of $t \in \mathbf{R}$, and a parameter $h > 0$ we put $A_h u(t) = \frac{1}{h}\int_{t-h}^t u(s)ds$. This is an approximation of u in L_2 and a continuous function (prove it!). Note, that the operator A_h retain the non-anticipation, because the averaging in the definition of A_h is over the past.

Proposition 6.2.4 *If $u \in L_2(\mathbf{R})$, then $A_h u \in L_2(\mathbf{R})$ and $A_h u \to u$ as $h \to 0$.*

Proof. The idea is to use the Fourier transform $\mathcal{F}u(x) = \frac{1}{\sqrt{2\pi}}\int e^{ixt}u(t)dt$. Since A_h is the convolution operator, we have $\mathcal{F}A_h u = b_h \mathcal{F}u$, where $b_h(y) = \frac{1}{h}\int_{-h}^0 e^{isy}ds = \frac{1-e^{ihy}}{ihy}$. Now, $\|u - A_h u\|_{L_2} = \|\mathcal{F}u - \mathcal{F}(A_h u)\|_{L_2}$, which is equal to $\|(1-b_h)\mathcal{F}u\|_{L_2} \to 0$ as $h \to 0$. Indeed, $b_h(y) = \frac{1-e^{ihy}}{ihy} \to 1$ as $h \to 0$

for any fixed y, and $|b_h(y)| \leq 1$. It remains to apply the Lebesgue bounded convergence theorem.

Now, since $g = A_h u$ is a continuous function, one can approximate g in L_2 as follows: $g_l(t) = g(2^{-l}[2^l t])$. Thus, in the limit, first $l \to \infty$, second $h \to 0$ we get the required approximation of u by simple functions.

At this point we can give the definition of the (Ito) stochastic integral for a general non-anticipating function.

Definition 6.2.5 *Suppose f is a non-anticipating function such that $\|f\|^2 := \mathbf{E}\int_0^t f(s)^2 ds$ is finite for any t, and f_n is an approximating sequence of simple non-anticipating functions so that $\|f - f_n\|^2 \to 0$. The Ito stochastic integral $\int_0^t f dw$ is defined as $\lim_{n\to\infty}\int_0^t f_n dw$. Here, $\int_0^t f_n dw$ is defined in (6.2.1), and $\lim_{n\to\infty}$ is taken in $L_2(\Omega)$, where Ω is the probability space of the white noise.*

Finally, the main result of the lecture is as follows:

Theorem 6.2.6 *The definition is correct and the defined integrals enjoys all properties 1–5 listed in Proposition 6.2.3.*

Proof is left as an exercise to the reader. (Warning! The statement 3 is difficult, and requires a technique we have not used yet.)

The reader should not think that the above definition of the stochastic integral is the only option. For instance, the integral sums

$$S_n = \sum_{k=0}^{[2^n t]-1} \frac{f_k + f_{k+1}}{2}\Delta_k \text{ or } \sum_{k=0}^{[2^n t]-1} f_{k+\frac{1}{2}}\Delta_k \tag{6.2.6}$$

lead to the (different!) Stratonovich integral, which is often denoted by

$$\int_0^t f(s) \circ dw(s) = \lim_{n\to\infty} S_n, \tag{6.2.7}$$

and which is better in some respects than the Ito one. However, Theorem 6.2.6 is wrong for the Stratonovich integrals. There exists, however, a simple relation between the Ito and the Stratonovich integrals.

Exercise 6.2.7 *Suppose, that $f(t) = f(0) + \int_0^t \phi(s)dw(s) + \int_0^t g(s)ds$, where ϕ and g are tempered non-anticipating functions. Prove that $\int_0^t f(s) \circ dw(s)$*

defined via (6.2.7) does exist, and is equal to $\int_0^t f(s)dw(s) + \frac{1}{2}\int_0^t \phi(s)ds$. *In other words,*

$$f \circ dw = fdw + \frac{1}{2}\phi\, dt$$

For example, $\int_0^t w(s) \circ dw(s) = \int_0^t w(s)dw(s) + \frac{1}{2}t$.

6.3 Extension of Stochastic Integral

This section is devoted to an extension of the notion of stochastic integral $\int f dw$ from tempered non-anticipating functions f, i.e., such that $\mathbf{E}\int |f|^2 dt < \infty$, to a wider class of functions f such that

$$\mathbf{P}\{\int |f|^2 dt < \infty\} = 1.$$

The main advantage of having this more general notion is that it allows to have Ito formulas without caring about unnecessary and ugly conditions like $\mathbf{E}\int_0^T \left|\sum_i u_i f_i\right|^2 (t, x(t))dt < \infty$. The extension comes with a price though: One cannot guarantee that $\mathbf{E}\int f dw = 0$, because the mathematical expectation may not exists.

6.3.1 The Doob inequality

Here, we prove an important inequality for stochastic integrals of non-anticipating functions. Suppose, $x(t) = \int_0^t f(s)dw(s)$ is a non-negative stochastic integral. The Doob inequality is about the probability of large surges of the function $x(t)$ on any bounded interval $I = [a, b]$ of time.

Theorem 6.3.1 *Suppose, that* $x(t) = x(0) + \int_0^t f(s)dw(s) \geq 0$ *is a non-negative (tempered) stochastic integral, and* $l > 0$ *is a constant. Then,*

$$\mathbf{P}\{\sup_{t\in I} x(t) \geq l\} \leq l^{-1}\mathbf{E}x(b). \tag{6.3.1}$$

The substance of the Doob inequality is in the estimation of the maximum of $x(t)$ over an *interval*. One can see easily, that for any *fixed* $t \in I$ we have $\mathbf{P}\{x(t) \geq l\} \leq l^{-1}\mathbf{E}x(t) = l^{-1}\mathbf{E}x(b)$, where the inequality is the Chebyshev

one, while the equality is one of the basic properties of stochastic integrals. Now we proceed to the actual proof.

Proof. Define the following time instant

$$\tau = \begin{cases} \inf\{t \in I : x(t) \geq l\}, \text{ if the set } \{t \in I : x(t) \geq l\} \text{ is not empty} \\ b, \text{ otherwise} \end{cases} \tag{6.3.2}$$

Consider the function $1_{<\tau}(t)$, which is 1 if $t < \tau$, and 0 otherwise. The crucial fact is that this is a non-anticipating function since its value at time t can be determined by the behavior of the process x up to this time. One says in this situation that τ is a *stopping time*, or *Markov moment*. Therefore, the stochastic integral

$$x(\tau) = x(0) + \int_0^\tau f(s)dw(s) := x(0) + \int_0^b 1_{<\tau}(s)f(s)dw(s)$$

is well-defined. Now we have, on the one hand, $\mathbf{E}x(\tau) = \mathbf{E}x(b)$ (explain why!), and, on the other hand,

$$\mathbf{P}\{\sup_{t\in I} x(t) \geq l\} = \mathbf{P}\{x(\tau) \geq l\}.$$

By the Chebyshev inequality we have $\mathbf{P}\{x(\tau) \geq l\} \leq l^{-1}\mathbf{E}x(\tau)$. Combining the Chebyshev inequality with two previous identities we arrive at the desired conclusion. ▶

Exercise 6.3.2 *Prove the Chebyshev inequality:* $\mathbf{P}\{\xi \geq l\} \leq l^{-1}\mathbf{E}\xi$, *where* $\xi \geq 0$ *is any nonnegative random variable.*

Exercise 6.3.3 *Prove the following extension of the Doob inequality:*

$$\mathbf{P}\{\sup_{t\in I} x(t) \geq l\} \leq l^{-1}\mathbf{E}x(b),$$

for $x(t) = x(0) + \int_0^t f(s)dw(s) + \int_0^t g(s)ds$, *where* $g \geq 0$.

6.3.2 Applications of the Doob inequality

Consider the stochastic integral $z(t) = e^{\int_0^t f(s)dw(s) - \frac{1}{2}\int_0^t |f(s)|^2 ds}$. By (tempered version of) the Ito formula, this is a stochastic integral indeed such that $z(t) = 1 + \int_0^t f(s)z(s)dw(s)$, at least in the case when $\mathbf{E}\int_0^t |f(s)z(s)|^2 ds < \infty$. The latter condition holds, say, when the integrand f is bounded. In what follows we will systematically truncate possibly unbounded functions f by using ${}_M f$ instead, where ${}_M f = f$ if $|f| \leq M$, and ${}_M f = 0$ otherwise.

Theorem 6.3.4 *Suppose that f is a bounded non-anticipating function, and consider the (tempered) stochastic integral $x(t) = \int_0^t f(s)dw(s)$. Then,*

$$\mathbf{P}\{\sup_{t\in I}\left(\int_0^t f(s)dw(s) - \frac{\alpha}{2}\int_0^t |f(s)|^2 ds\right) \geq \beta\} \leq e^{-\alpha\beta}, \tag{6.3.3}$$

where α, β are arbitrary positive constants.

This immediately follows from the Doob inequality (6.3.1) as applied to the positive stochastic integral $z_\alpha(t) = e^{\int_0^t \alpha f(s)dw(s) - \frac{1}{2}\int_0^t |\alpha f(s)|^2 ds}$. ▶

Now, consider a sequence of bounded non-anticipating functions f_n such that

$$\mathbf{P}\left[\int_0^1 |f_n(s)|^2 ds \leq 2^{-n},\ n \uparrow \infty\right] = 1. \tag{6.3.4}$$

Here, the notation $\mathbf{P}\,[A_n,\ n \uparrow \infty]$ stands for the probability of the event $\bigcup_{k\geq 1}\bigcap_{n\geq k} A_n$, meaning that all the events A_n hold as soon as n is sufficiently large. Fix a constant $\theta > 1$, and consider the function $h(x) = \sqrt{2x\log|\log x|}$. Note that $h(2^{-n}) = \sqrt{2^{-n+1}\log n}$. Then,

$$\mathbf{P}\left\{\sup_{t\in I}\left|\int_0^t f_n dw\right| \leq \theta h(2^{-n}),\ n \uparrow \infty\right\} = 1. \tag{6.3.5}$$

Indeed, consider the inequality (6.3.3), where $f = f_n$, $\alpha = (2^{n+1}\log n)^{1/2}$, $\beta = \theta(2^{-n-1}\log n)^{1/2}$. We obtain

$$\mathbf{P}\{\sup_{t\in I}\int_0^t f_n dw \geq \frac{\alpha}{2}\int_0^t |f_n|^2 ds + \beta\} \leq n^{-\theta}$$

which implies (explain why!) that

$$\mathbf{P}\{\sup_{t\in I}\int_0^t f_n dw \leq \frac{\alpha}{2}\int_0^t |f_n|^2 ds + \beta,\ n \uparrow \infty\} = 1.$$

Note now, that $\frac{\alpha}{2}2^{-n} + \beta = \frac{1+\theta}{2}h(2^{-n})$, and in view of (6.3.4) we obtain

$$\mathbf{P}\left\{\frac{\alpha}{2}\int_0^t |f_n|^2 ds + \beta \leq \frac{1+\theta}{2}h(2^{-n}),\ n \uparrow \infty\right\} = 1,$$

and

$$\mathbf{P}\left\{\sup_{t\in I}\int_0^t f_n dw \leq \frac{1+\theta}{2}h(2^{-n}),\ n \uparrow \infty\right\} = 1.$$

The same arguments prove that

$$\mathbf{P}\left\{\sup_{t\in I}\int_0^t(-f_n)dw \le \frac{1+\theta}{2}h(2^{-n}),\ n\uparrow\infty\right\}=1.$$

Summing up, we obtain

$$\mathbf{P}\left\{\sup_{t\in I}\left|\int_0^t f_n dw\right| \le \frac{1+\theta}{2}h(2^{-n}),\ n\uparrow\infty\right\}=1$$

which is just another incarnation of (6.3.5).

Now, we can easily extend the stochastic integral $\int_0^t f dw$ to any non-anticipating function f such that $\mathbf{P}\left[\int_0^t |f(s)|^2 ds<\infty\right]=1$. Indeed, for any given n one can find M such that $\mathbf{P}A_n \le 2^{-n}$, where the event A_n is

$$A_n=\left\{\int_0^t |f(s)-_M f(s)|^2 ds \ge 2^{-n-1}\right\},$$

for $\int_0^t |f(s)-_M f(s)|^2 ds\to 0$ as $M\to\infty$. Put $g_n =_M f$, and $f_n = g_{n+1}-g_n$. We have $\int_0^t |f_n|^2 ds \le 2(\int_0^t |f-g_n|^2 ds + \int_0^t |f-g_{n+1}|^2 ds)$. Therefore,

$$\mathbf{P}\left[\int_0^t |f_n|^2 ds \ge 2^{-n+1}\right] \le \mathbf{P}A_n + \mathbf{P}A_{n+1} \le 2^{-n+1}.$$

From (6.3.5) we now conclude, that for (almost) any sample path of the white noise we have $\sup_{t\in I}\left|\int_0^t f_n dw\right| \le \theta h(2^{-n+1})$ as soon as n is sufficiently large. But $f = g_0 + \sum_{n=1}^\infty f_n$, and the series $\sum_{n=1}^\infty \int_0^t f_n dw$ converges absolutely, and uniformly in I. Thus, we might unambiguously put

$$\int_0^t f dw := \int_0^t g_0 dw + \sum_{n=1}^\infty \int_0^t f_n dw \tag{6.3.6}$$

which gives our final definition of the stochastic integral. It is automatically a continuous function of the upper limit.

Exercise 6.3.5 *Prove (a part of) the iterated logarithm law:*

$$\mathbf{P}\left[\sup_{t\in[0,T]}|w(t)| \le \theta h(T),\ T\uparrow\infty\right]=1,$$

where θ is an arbitrary constant >1, and $h(T)=(2T\log\log T)^{1/2}$.

Hint: Use the same method as used in proving (6.3.5).

The full iterated logarithm law is the following statement:

Theorem 6.3.6 *The set of limit values of* $w(T)/h(T)$ *as* $T \to \infty$ *coincides with the interval* $[-1, 1]$ *for almost every sample path of the Wiener process.*

It is highly nontrivial to prove, but accessible with the above technique. First proved by outstanding Russian mathematician A.Ya. Khinchin in 1923.

6.4 Ito's formula

6.4.1 Statement of the Ito lemma

Now we come to the main formula of the stochastic calculus which is an analog of the chain rule $(f(g(t))' = f'(g(t))g'(t))$ of the ordinary calculus. It is, however, much deeper than the deterministic analog.

First, introduce a convenient notation: Consider the stochastic non-anticipating process

$$x(t) = x(0) + \int_0^t f(s)dw(s) + \int_0^t g(s)ds,\ f, g \in \mathbf{R}^n \tag{6.4.1}$$

We are going to write instead of (6.4.1)

$$dx(t) = f(t)dw(t) + g(t)dt, \tag{6.4.2}$$

so that the Newton–Leibnitz formula $d\int \omega = \omega$, where $\omega = f(t)dw(t) + g(t)dt$, holds by definition.

By using coordinates, (6.4.1) might be rewritten as $dx_i = f_i dw + g_i dt$.

Similar, but different notion of the differential is associated to the Stratonovich integral, so that we will write

$$dx = f \circ dw + gdt \tag{6.4.3}$$

instead of

$$x(t) = x(0) + \int_0^t f(s) \circ dw(s) + \int_0^t g(s)ds. \tag{6.4.4}$$

This suggests the following question. Suppose, u is a smooth function. What is the differential of $u(x(t))$? In other words, is the class of non-anticipating processes represented as the Ito integrals closed under superposition with f, and, if this is the case, how to represent $u(x(t))$ as a stochastic integral?

Note that the process x is, generally speaking, vector valued, so u is defined on a finite-dimensional vector space.

Theorem 6.4.1 (Ito formula) *Suppose, that a function $u(t,x)$, where $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, is such that the partial derivatives*

$$u_t = \frac{\partial u}{\partial t},\ u_i = \frac{\partial u}{\partial x_i},\ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

are continuous on $\mathbf{R} \times \mathbf{R}^n$. Then, the superposition $u(t) = u(t, x(t))$, where x is the stochastic integral (6.4.1) is again a stochastic integral, and its differential is given by

$$du = u_t dt + \sum_{i=1}^{n} u_i dx_i + \frac{1}{2} \sum_{i,j=1}^{n} u_{ij} dx_i dx_j. \tag{6.4.5}$$

Here, $u_t = u_t(t, x(t))$, $u_i = u_i(t, x(t))$, $u_{ij} = u_{ij}(t, x(t))$, $dx_i dx_j := f_i f_j dt$.

In fact, at the moment, when we have had defined the Ito integral only for *tempered* non-anticipating function, the above statement is not absolutely true, for the integrand like $\sum_{i=1}^{n} u_i dx_i$ might not be tempered. Still the above Ito formula is valid, but requires an extension of the notion of the stochastic integral, which we postpone. The "tempered" version of the Ito formula is this:

Theorem 6.4.2 (Tempered Ito formula) *Suppose, that a function $u(t,x)$ is such that the partial derivatives*

$$u_t = \frac{\partial u}{\partial t},\ u_i = \frac{\partial u}{\partial x_i},\ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

are continuous on $\mathbf{R} \times \mathbf{R}^n$, and the mathematical expectation

$$\mathbf{E} \int_0^T \left| \sum_i u_i f_i \right|^2 (t, x(t)) dt$$

is finite for any finite T. Then, the superposition $u(t) = u(t, x(t))$, where x is the tempered stochastic integral (6.4.1) is again a tempered stochastic integral, and its differential is given by

$$du = u_t dt + \sum_{i=1}^{n} u_i dx_i + \frac{1}{2} \sum_{i,j=1}^{n} u_{ij} dx_i dx_j. \tag{6.4.6}$$

Here, $u_t = u_t(t, x(t))$, $u_i = u_i(t, x(t))$, $u_{ij} = u_{ij}(t, x(t))$, $dx_i dx_j := f_i f_j dt$.

In what follows we will not dwell on "tempered" versions of theorems, which become valid without this assumption after suitable extension of the notion of stochastic integral.

The main difference between the usual and the stochastic calculus is the presence of the summand $\frac{1}{2}\sum_{i,j=1}^{n} u_{ij}dx_i dx_j$ in (6.4.5). Its appearance can be heuristically explained as follows: If $\Delta^h = x(t+h) - x(t)$, then $\Delta_i^h \Delta_j^h$ has order of magnitude h, not h^2 as in the deterministic setup. For example, $\mathbf{E}(w(t+h) - w(t))^2 = h$.

However, in the Stratonovich case, the stochastic calculus takes a familiar form:

Theorem 6.4.3 (Ito–Stratonovich formula) *Suppose, that a function $u(t,x)$, where $t \in \mathbf{R}$, $x \in \mathbf{R}^n$ enjoys the same properties as in the previous theorem. Then, the superposition $u(t) = u(t, x(t))$, where x is the stochastic integral (6.4.1) is again a stochastic (Stratonovich) integral, and its differential is given by*

$$du = u_t dt + \sum_{i=1}^{n} u_i \circ dx_i. \tag{6.4.7}$$

Of course the above two forms of the chain rule are compatible:

Exercise 6.4.4 *Deduce the Ito–Stratonovich formula from the Ito formula.*

(Hint: $\int u_i \circ dx_i = \int u_i dx_i + \frac{1}{2} du_i dx_i = \int u_i dx_i + \frac{1}{2}\sum_j u_{ij} dx_i dx_j$ because of the definition of the Stratonovich integral, and the Ito formula applied to u_i)

6.4.2 Vector version of the Ito formula

We are going to use widely the rather obvious vector counterpart of (6.4.2) where $w(t) \in \mathbf{R}^m$ is the vector-valued Wiener process, meaning that $w(t) = (w_1(t), \ldots, w_m(t)) \in \mathbf{R}^m$, where w_i are *independent* scalar Wiener processes.

In other words, in the equation

$$dx(t) = f(t)dw(t) + g(t)dt, \tag{6.4.8}$$

this time, $f(t)$ is $n \times m$ matrix, and not a vector as in (6.4.2). Then, the vector Ito formula takes the following form:

Theorem 6.4.5 *Suppose, that a function $u(t,x)$, where $t \in \mathbf{R}$, $x \in \mathbf{R}^n$ enjoys the same properties as in theorem 6.4.1. Then, $u(t) = u(t,x(t))$, where x is the stochastic integral (6.4.8) is again a stochastic integral, and its differential is given by*

$$du = u_t dt + \sum_{i=1}^{n} u_i dx_i + \frac{1}{2}\sum_{i,j=1}^{n} u_{ij} dx_i dx_j. \tag{6.4.9}$$

Here, $u_t = u_t(t,x(t))$, $u_i = u_i(t,x(t))$, $u_{ij} = u_{ij}(t,x(t))$, $dx_i dx_j := \langle f_i, f_j\rangle dt$, where the scalar product of rows f_i, f_j of the matrix f is given by $\langle f_i, f_j\rangle := \sum_{k=1}^{m} f_{ik} f_{jk}$.

In other words, the multiplication table for stochastic differentials is

$$dw_i dw_j = \delta_{ij} dt,\ dw_i dt = 0 = (dt)^2.$$

Again, the Stratonovich version is this: The governing equation is

$$dx(t) = f(t) \circ dw(t) + g(t)dt, \tag{6.4.10}$$

and then

$$du = u_t dt + \sum_{i=1}^{n} u_i \circ dx_i. \tag{6.4.11}$$

Exercise 6.4.6 *Deduce the Stratonovich formula (6.4.11) from the Ito formula (6.4.9).*

6.4.3 First applications of the Ito formula

We present some important corollaries of the Ito formula via a sequence of exercises.

Exercise 6.4.7 *Show that stochastic integrals are closed under multiplication: if $dx(t) = f(t)dw(t) + g(t)dt$, $dy(t) = h(t)dw(t) + k(t)dt$, then $d(xy) = xdy + ydx + dxdy$, where $dxdy = fhdt$.*

(Hint: apply the Ito formula to $u(x,y) = xy$.)

Exercise 6.4.8 *Find $\int_0^t w dw$ and $\int_0^t w \circ dw$.*

(Answer: $\int_0^t w dw = \frac{1}{2}w(t)^2 - \frac{1}{2}t$, $\int_0^t w \circ dw = \frac{1}{2}w(t)^2$. Hint: apply the Ito formula to $u(x) = \frac{1}{2}x^2$)

Exercise 6.4.9 *Find* $\mathbf{E}\exp\left(\int_0^t f dw - \frac{1}{2}\int_0^t f^2 ds\right)$.

(Answer: $\mathbf{E}\exp\left(\int_0^t f dw - \frac{1}{2}\int_0^t f^2 ds\right) = 1$. Hint: apply the Ito formula to $x(t) = \int_0^t f dw - \frac{1}{2}\int_0^t f^2 ds$, and $u(x) = \exp x$ in order to show that $du = f u dw$. Then, apply the basic property of stochastic integrals: $\mathbf{E}\int g dw = 0$ for any g.)

Define the Hermite polynomials $H_n(t,x)$ via the generating function

$$\sum_{n=0}^{\infty} z^n H_n = \exp\left(zx - \frac{1}{2}z^2 t\right). \tag{6.4.12}$$

Exercise 6.4.10 *Show that* $H_n(t,x) = \frac{(-t)^n}{n!}\exp(x^2/2t)\frac{\partial^n}{\partial x^n}\exp(-x^2/2t)$.

Hint: $\exp\left(zx - \frac{1}{2}z^2 t\right) = \exp(x^2/2t)\exp\left(-\frac{(x-tz)^2}{2t}\right)$

Exercise 6.4.11 *Prove that*

$$H_n(t, w(t)) = \int_0^t dw(t_1)\int_0^{t_1} dw(t_2)\ldots\int_0^{t_{n-1}} dw(t_n). \tag{6.4.13}$$

Note that (6.4.13) might be symbolically expressed as

$$n! H_n(t, w(t)) = \int_0^t \int_0^t \ldots \int_0^t dw(t_1) dw(t_2)\ldots dw(t_n). \tag{6.4.14}$$

Hint: This is equivalent to

$$\sum_{n=0}^{\infty} z^n \int_0^t dw(t_1)\int_0^{t_1} dw(t_2)\ldots\int_0^{t_{n-1}} dw(t_n) = \exp\left(zw(t) - \frac{1}{2}z^2 t\right). \tag{6.4.15}$$

In order to prove (6.4.15) show that both sides satisfy the same stochastic differential equation $du = zu dw$. Then apply the standard existence and uniqueness theorem for stochastic differential equations (to be lectured later).

Another approach is to derive from (6.4.12) that

$$\frac{\partial H_n}{\partial t} + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2} = 0, \text{ and } \frac{\partial H_n}{\partial x} = H_{n-1}.$$

Then apply induction w.r.t. index n.

Exercise 6.4.12 *Prove that*

$$\frac{1}{\sqrt{2\pi t}} \int H_n(t,x) H_m(t,x) e^{-\frac{x^2}{2t}} dx = \frac{\delta_{mn}}{n!}. \tag{6.4.16}$$

Hint: $\int H_n(t,x) H_m(t,x) e^{-\frac{x^2}{2t}} dx = \mathbf{E} H_n(t,w(t)) H_m(t,w(t))$. Then, use the result of the previous exercise together with the basic properties of the stochastic integrals.

Another approach is to use

$$\sum_{n,m} z^n w^m \int H_n(t,x) H_m(t,x) e^{-\frac{x^2}{2t}} dx = \int e^{(z+w)x - \frac{1}{2}(z^2+w^2)t} e^{-\frac{x^2}{2t}} dx =$$

$$= e^{tzw} \int e^{(z+w)x - \frac{1}{2}(z^2+w^2+2zw)t} e^{-\frac{x^2}{2t}} dx = e^{tzw} \int e^{\frac{(x-(z+w)t)^2}{2t}} dx = e^{tzw} \sqrt{2\pi t}$$

Exercise 6.4.13 *Prove that*

$$\sum_{s=0}^{n} H_s(t,x) H_{n-s}(t,y) = 2^{n/2} H_n(t, \frac{x+y}{\sqrt{2}}) \tag{6.4.17}$$

Hint: Put $w(t) = \frac{w'(t)+w''(t)}{\sqrt{2}}$, where $w'(t)$, $w''(t)$ are independent Wiener processes. Then, $w(t)$ is a Wiener process. Apply (6.4.14).

Exercise 6.4.14 *Suppose, $w(t) = w'(t) + iw''(t) \in \mathbf{C}$, where w', w'' are scalar (real) independent Wiener processes. Let $f(z)$ be an entire (everywhere holomorphic) function. Then $f(w(t)) = W(\tau(t))$, where W is a complex valued Wiener process, and $\tau(t) = \int_0^t |f'(w(s))|^2 ds$ is a new time scale, where f' is the complex derivative of f.*

Hint: Prove first that $df(w(t)) = f'(w(t)) dw(t)$.

6.4.4 Proof of the (tempered) Ito formula

We shall confine ourself with proving the Ito formula in case $x(t) = w(t)$, and $u(t,x) = u(x)$ has uniformly bounded 3-rd differential. The general case does not require new ideas in order to conduct a proof.

We also consider only the case $t = 1$ for simplicity. Then, the result to be proved is

$$u(w(1)) - u(0) = \int_0^1 \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(w(t))dt + \int_0^1 \frac{\partial u}{\partial x}(w(t))dw(t). \tag{6.4.18}$$

The temperedness assumption boils down to $\mathbf{E}\int_0^1 \left|\frac{\partial u}{\partial x}\right|^2 (w(t))dt < \infty$. The idea is to write down

$$u(w(1)) - u(0) = \sum_{k=0}^{2^n-1}(u_{k+1} - u_k), \tag{6.4.19}$$

where $u_k = u(w(\frac{k}{2^n}))$, and

$$u_{k+1} = u_k + \left(\frac{\partial u}{\partial x}\right)_k \Delta_k + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_k \Delta_k^2 + R_k, \tag{6.4.20}$$

where $\Delta_k = w(\frac{k+1}{2^n}) - w(\frac{k}{2^n})$, and the remainder $R_k \le M|\Delta_k|^3$ because of our assumption on boundedness of the 3-rd differential of u.

Now, it remains to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(\frac{\partial u}{\partial x}\right)_k \Delta_k = \int_0^1 \frac{\partial u}{\partial x}(w(t))dw(t), \tag{6.4.21}$$

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(\frac{\partial^2 u}{\partial x^2}\right)_k \Delta_k^2 = \int_0^1 \frac{\partial^2 u}{\partial x^2}(w(t))dt, \tag{6.4.22}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} R_k = 0. \tag{6.4.23}$$

Here, all the limits are to be understood in the mean square sense.

We notice immediately, that (6.4.21) holds by the definition of the stochastic integral. To prove (6.4.23), we note that $\|\sum_{k=0}^{2^n-1} R_k\| \le 2^n \max_k \|R_k\|$, while $\|R_k\| \le C2^{-\frac{3}{2}n}$. Here, $\|\|$ stands for the L_2-norm, and C is an absolute constant. Thus, $\|\sum_{k=0}^{2^n-1} R_k\| \le C2^{-\frac{1}{2}n}$, and (6.4.23) is proved.

To prove (6.4.22) we note that by definition

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(\frac{\partial^2 u}{\partial x^2}\right)_k 2^{-n} = \int_0^1 \frac{\partial^2 u}{\partial x^2}(w(t))dt, \tag{6.4.24}$$

and it remains to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(\frac{\partial^2 u}{\partial x^2}\right)_k (\Delta_k^2 - 2^{-n}) = 0. \tag{6.4.25}$$

Put $v = \frac{\partial^2 u}{\partial x^2}$, $\phi_k = \Delta_k^2 - 2^{-n}$. We have

$$\mathbf{E}\left|\sum_{k=0}^{2^n-1} v_k \phi_k\right|^2 = \sum_{k=0}^{2^n-1} \mathbf{E}|v_k|^2 \mathbf{E}|\phi_k|^2 \tag{6.4.26}$$

because of non-anticipation property of v, and the fact that $\mathbf{E}\phi_k = 0$. The right-hand side of (6.4.26) has the form $C2^{-2n}\sum_{k=0}^{2^n-1}\mathbf{E}|v_k|^2$ and tends to zero.

Exercise 6.4.15 *The above statement is based on the inequality*

$$\mathbf{E}|v_k|^2 \le C \tag{6.4.27}$$

where C does not depend on k. Prove (10.1.4).

Hint: $v_k = \phi(x(\frac{k}{2^n}))$, where ϕ is a Lipschitz continuous function. This reduces our task to proving the inequality

$$\mathbf{E}|x(\frac{k}{2^n})|^2 \le C. \tag{6.4.28}$$

This, in turn, follows from (6.4.2) and the basic property

$$\mathbf{E}\left|\int f dw\right|^2 = \mathbf{E}\int |f(s)|^2 ds$$

of stochastic integrals.

6.5 Stochastic Differential Equations

6.5.1 The Cauchy Problem

This is the following problem: We are given formal equation and initial condition

$$dx(t) = f(t, x(t))dw(t) + g(t, x(t))dt, \quad x(0) = \eta, \tag{6.5.1}$$

where $x(t) \in V = \mathbf{R}^n$, η is a random vector of V, independent of the white noise $\xi(t) = \dot{w}(t)$ for $t > 0$, $w(t) \in W = \mathbf{R}^m$ is the m-dimensional Wiener process, $f(t,x)$ is a $n \times m$-matrix, $g(t,x) \in V$. The equation (14.1.15) is called stochastic differential equation (SDE), or rather the Cauchy Problem for SDE. The solution $x(t)$ is, by definition, a non-anticipating function such that

$$x(t) - \eta = \int_0^t f(s, x(s))dw(s) + \int_0^t g(s, x(s))ds. \tag{6.5.2}$$

In more advanced courses this notion is called *strong solution* in order to distinguish from *weak solution*, a notion we are not going to deal with. In fact, the reasons for introduction of weak solutions are quite sound. Primarily, this is because known results on existence and uniqueness of strong solutions are not always adequate.

Still, here, we confine ourself with stating and proving a direct analog of the Cauchy–Lipschitz theorem for Ordinary Differential Equations.

Theorem 6.5.1 *Suppose that the functions f, g are non-anticipating functions of t, which are Lipshitz w.r.t. x. Suppose also that the following bound holds: The (squares of) L_2-norms*

$$\mathbf{E}|\eta|^2, \int_0^T \mathbf{E}|a(t,\eta)|^2 dt, \int_0^T \mathbf{E}|b(t,\eta)|^2 dt \text{ are bounded} \tag{6.5.3}$$

for any fixed finite T. Then, there exists a unique (strong) solution to the Cauchy problem (14.1.15).

Corollary 6.5.2 *In particular, if $f = f(t)$ does not depend on x, while g has the form $g(t,x) = A(t)x$, where $A(t)$ is a matrix, then, there exists a unique (strong) solution to the Cauchy problem (14.1.15). Moreover, the solution is given by the Cauchy formula:*

$$x(t) = \Phi(t,0)\eta + \int_0^t \Phi(t,s)f(s)dw(s), \tag{6.5.4}$$

where $\Phi(t,s)$ is the fundamental matrix, i.e., solution to

$$\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s),\ \Phi(s,s) = \mathrm{id}$$

Exercise 6.5.3 *Find the differential equation satisfied by $\Phi(t,s)$ as the function of s.*

Exercise 6.5.4 *Prove corollary 6.5.2.*

Similarly, one can define solutions to the Stratonovich counterpart of (14.1.15)

$$dx(t) = f(t, x(t)) \circ dw(t) + g(t, x(t))dt, x(0) = \eta, \tag{6.5.5}$$

which is equivalent to the Ito equation

$$dx_i(t) = f_{ij}(t, x(t))dw_j(t) + \left(\frac{1}{2}\frac{\partial f_{ij}}{\partial x_k} f_{kj}(t, x(t)) + g_i(t, x(t))\right) dt, x(0) = \eta, \tag{6.5.6}$$

where summation is tacitly performed over repeated indices.

Exercise 6.5.5 *Give a precise definition of the solution to the Stratonovich equations, and explain why equations (6.5.6) and (6.6.7) are equivalent.*

Thus, the standard existence and uniqueness theorem for the Stratonovich equations involves Lipshitz bounds on *derivatives* of f. Still, in many situations, the Stratonovich equations are more relevant than the Ito ones.

6.5.2 Proof of the Cauchy–Lipschitz theorem for SDE

We begin with the uniqueness clause. Suppose that $x(t)$, $y(t)$ are two strong solutions to (14.1.16). Then, we get the following equation for the difference $z(t) = x(t) - y(t)$:

$$z(t) = \int_0^t (f(s, x(s)) - f(s, y(s)))\, dw(s) + \int_0^t (g(s, x(s)) - g(s, y(s)))\, ds, \tag{6.5.7}$$

and for the variance $\phi(t) = \mathbf{E}|z(t)|^2$ we obtain, by using inequality $|A+B|^2 \leq 2(|A|^2 + |B|^2)$ (prove it!), that

$$\begin{aligned} \phi(t) \leq 2\mathbf{E} \left| \int_0^t (f(s, x(s)) - f(s, y(s)))\, dw(s) \right|^2 + \\ +2\mathbf{E} \left| \int_0^t (g(s, x(s)) - g(s, y(s)))\, ds \right|^2. \end{aligned} \tag{6.5.8}$$

Now, by basic properties of stochastic integrals

$$\mathbf{E} \left| \int_0^t (f(s, x(s)) - f(s, y(s)))\, dw(s) \right|^2 = \mathbf{E} \int_0^t |f(s, x(s)) - f(s, y(s))|^2\, ds, \tag{6.5.9}$$

and by the Lipschitz condition

$$\mathbf{E}\int_0^t |f(s,x(s)) - f(s,y(s))|^2\, ds \le C\int_0^t \phi(s)ds. \tag{6.5.10}$$

Similarly, by the Cauchy inequality

$$\begin{aligned}\mathbf{E}\left|\int_0^t (g(s,x(s)) - g(s,y(s)))\, ds\right|^2 &\le t\mathbf{E}\int_0^t |g(s,x(s)) - g(s,y(s))|^2\, ds \\ &\le tC\int_0^t \phi(s)ds.\end{aligned} \tag{6.5.11}$$

We obtain for the positive function ϕ on any bounded time interval the inequality of the form

$$\phi(t) \le C\int_0^t \phi(s)ds, \tag{6.5.12}$$

where C is a constant. Now, the Gronwall inequality proves that $\phi \equiv 0$, and the uniqueness part is established.

Exercise 6.5.6 *Prove the following Gronwall inequality: Suppose that a non-negative function $\phi(t)$ satisfies*

$$\phi(t) \le A + C\int_0^t \phi(s)ds \text{ if } 0 \le t \le T. \tag{6.5.13}$$

Then,

$$\phi(t) \le A\exp(Ct) \text{ if } 0 \le t \le T. \tag{6.5.14}$$

To prove existence, we use the Picard iteration scheme:

$$x_0(t) \equiv \eta, \tag{6.5.15}$$

and

$$x_{n+1}(t) - \xi = \int_0^t f(s,x_n(s))dw(s) + \int_0^t g(s,x_n(s))ds. \tag{6.5.16}$$

Then, for the differences $z_n = x_{n+1} - x_n$ we obtain an analog of (6.5.7):

$$\begin{aligned} z_n(t) \;=\; &\int_0^t (f(s,x_{n+1}(s)) - f(s,x_n(s)))\, dw(s) + \\ &\int_0^t (g(s,x_{n+1}(s)) - g(s,x_n(s)))\, ds, \end{aligned} \tag{6.5.17}$$

and for their variances $\phi_n(t) = \mathbf{E}|z_n(t)|^2$ by the same arguments as in the uniqueness part we get the inequality

$$\phi_{n+1}(t) \leq C \int_0^t \phi_n(s) ds, \tag{6.5.18}$$

valid in any given bounded interval of time $[0, T]$. Now, (6.5.18) gives by induction that

$$\phi_n(t) \leq A \frac{C^n t^n}{n!}. \tag{6.5.19}$$

Note, that the base of induction is provided by condition (6.5.3).

Exercise 6.5.7 *Prove the following inequality: Suppose that a sequence of non-negative functions $\phi_n(t)$ satisfies (6.5.18). Then, (6.5.19) holds (with some constants A and C).*

The basic inequality (6.5.19) proves that the sequence x_n converges uniformly in any given bounded interval of time as the sequence of continuous functions on $[0, \infty)$ with values in square integrable random variables. This allows us to pass to the limit in the equation (6.5.16), thus completing the existence proof.

Exercise 6.5.8 *Conduct detailed proof of the existence of $x = \lim_{n\to\infty} x_n$, and the equality*

$$x(t) - \eta = \int_0^t f(s, x(s)) dw(s) + \int_0^t g(s, x(s)) ds.$$

6.5.3 Dependence on parameters

Here, the basic result is as follows: Suppose that the conditions of the Cauchy-Lipschitz theorem 6.5.1 are met, and, moreover, the coefficients f, g are smooth functions of the space variable x, depending smoothly on a parameter $\epsilon \in (-\epsilon_0, \epsilon_0)$. Similarly, the initial condition $\eta = \eta(\epsilon)$ is smooth w.r.t. ϵ. Then, the corresponding solution $x(t; \epsilon)$ of the Cauchy problem smoothly depends on ϵ as well. Moreover, the derivative $X(t) = \frac{\partial}{\partial \epsilon} x(t; \epsilon)$ satisfies *linear* SDE with non-anticipating coefficients

$$\begin{aligned} dX &= \left(\frac{\partial f}{\partial x}(t, x(t)) X + \frac{\partial f}{\partial \epsilon}(t, x(t)) \right) dw + \left(\frac{\partial g}{\partial x}(t, x(t)) X + \frac{\partial f}{\partial \epsilon}(t, x(t) \right) dt, \\ X(0) &= H, \end{aligned} \tag{6.5.20}$$

where $H = \frac{\partial}{\partial \epsilon}\eta(\epsilon)$. By using indices we can rewrite (6.5.20) in a more explicit form:

$$\begin{aligned} dX_i &= \sum_{k=1}^m \left(\tfrac{\partial f_{ik}}{\partial \epsilon}(t, x(t)) + \sum_{l=1}^n \tfrac{\partial f_{ik}}{\partial x_l}(t, x(t)) X_l \right) dw_k + \\ &\quad \sum_{l=1}^n \tfrac{\partial g_i}{\partial x_l}(t, x(t)) X_l dt + \tfrac{\partial g_i}{\partial \epsilon}(t, x(t)) dt, \qquad (6.5.21) \\ X(0) &= H, \end{aligned}$$

To prove (6.5.20) one might consider the equations (14.1.15), (6.5.20) as a *system* of SDE for $\mathcal{X} = (x, X)$. The Cauchy problem for $\mathcal{X}$ has a unique solution by Theorem 6.5.1. Moreover, this solution is the limit of Picard iterations $\mathcal{X}_n = (x_n, X_n)$, and at each step of the iteration process we have

$$X_n(t; \epsilon) = \frac{\partial}{\partial \epsilon} x_n(t; \epsilon). \qquad (6.5.22)$$

This implies that $x = \lim x_n$ is differentiable w.r.t. ϵ, and the derivative is given by $X = \lim X_n$ (provide detail!).

An important example of SDE with a parameter is

$$dx(t) = f(t, x(t))dw(t) + g(t, x(t))dt, x(0) = x, \qquad (6.5.23)$$

where the deterministic initial vector x serve as a parameter. From the preceding arguments we conclude that the map $x \mapsto x(t)$, where the time $t \geq 0$ is fixed, is a smooth map, in fact, a diffeomorphism. Note, that the map $t \mapsto x(t)$ is rarely smooth, e.g., the brownian path is nowhere differentiable. As an application, consider a smooth function ϕ and the function $x \mapsto \mathbf{E}\phi(x(t))$. This function is smooth. In fact, the same is true for the function $t \mapsto \mathbf{E}\phi(x(t))$ in spite of the mentioned fact that the diffusion prevents smoothness of the sample path $t \mapsto x(t)$. Indeed, consider the stochastic differential of $\phi(x(t))$. We obtain, according to the Ito formula,

$$\frac{d}{dt}\mathbf{E}\phi(x(t)) = \mathbf{E}\frac{d}{dt}\phi(x(t)) = \mathbf{E}(\mathcal{L}\phi)(x(t)), \qquad (6.5.24)$$

where $\mathcal{L}$ is the second order partial differential operator

$$\mathcal{L} \stackrel{\text{def}}{=} \sum_i g_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{ijk} f_{ik} f_{jk} \frac{\partial^2}{\partial x_i \partial x_j}. \qquad (6.5.25)$$

Since ϕ is smooth, the right-hand side of (6.5.24) is a continuous function of t, and thus, the function $t \mapsto \mathbf{E}\phi(x(t))$ is C^1-smooth. By iteration of the above arguments we get the desired C^∞-smoothness of the function.

6.6 Diffusion Processes

Diffusion Process i.e. Infinitesimal Operator is the same thing as solution to a stochastic differential equation

$$dx(t) = f(t, x(t))dw(t) + g(t, x(t))dt, x(s) = x, \tag{6.6.1}$$

Here, s, resp. x is initial time, resp. initial position of the process. We assume in what follows that the coefficients $f(t, x)$, which is a matrix, and $g(t, x)$, which is a vector, are sufficiently smooth, so that, in particular, the Lipshitz condition from the standard existence & uniqueness theorem of the previous lecture holds.

Note, that if $f \equiv 0$ we are talking about deterministic motion along trajectories of an ordinary differential equation. It is well-known that in this case there is a close connection between this dynamic system

$$\dot{x}(t) = g(t, x(t)), x(s) = x, \tag{6.6.2}$$

and the partial differential equation

$$\frac{\partial u}{\partial s}(s, x) + \sum_i g_i \frac{\partial u}{\partial x_i}(s, x) = 0, \tag{6.6.3}$$

which is dual to the equation

$$\frac{\partial \rho}{\partial t} = \div b\rho \tag{6.6.4}$$

describing evolution of the density of particles moved by the phase flow of (6.6.2). If the vector field b is sufficiently regular, say, if the Cauchy-Lipschitz condition holds, one can write down the general solution of (6.6.3) in terms of the phase flow. Namely, suppose that the function $\phi(x) = u(T, x)$ is known for some T. Then,

$$u(s, x) = \phi(x(T)), \tag{6.6.5}$$

where $x(t)$ is the solution of (6.6.2). It is very easy to check (6.6.5) at least at the formal level. Indeed, consider the function $t \mapsto u(t, x(t))$. Its derivative is

$$\frac{\partial u}{\partial s}(t, x(t)) + \sum_i g_i(t, x(t)) \frac{\partial u}{\partial x_i}(t, x(t)),$$

which is zero by (6.6.3). Therefore, $u(s,x) = u(x(T),T) = \phi(x(T))$. Now we indicate the counterparts of these facts for SDE (14.1.15). To do this we will use a very important second order partial differential operator

$$\mathcal{L} \stackrel{\text{def}}{=} \sum_i g_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{ijk} f_{ik} f_{jk} \frac{\partial^2}{\partial x_i \partial x_j}, \tag{6.6.6}$$

which is called the *infinitesimal* operator or *generator* of the diffusion process (14.1.15). We indicate immediately, that for the Stratonovich equation

$$dx(t) = f(t,x(t)) \circ dw(t) + g(t,x(t))dt, x(s) = x, \tag{6.6.7}$$

the infinitesimal operator takes the form

$$\mathcal{L} = \sum_i g_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_k \left(\sum_i f_{ik} \frac{\partial}{\partial x_i} \right)^2. \tag{6.6.8}$$

In (6.6.6), (6.6.8) the indices $i, j \in [1,n]$, $k \in [1,m]$. We need also the *adjoint* operators $\mathcal{L}^*$, defined by the identity

$$\int \mathcal{L}u(x)v(x)dx = \int u(x)\mathcal{L}^* v(x)dx,$$

where the functions u, v are smooth and have compact support. The formula for $\mathcal{L}^*$ has the form

$$\mathcal{L}^* = -\sum_i \frac{\partial}{\partial x_i} g_i + \frac{1}{2} \sum_{ijk} \frac{\partial^2}{\partial x_i \partial x_j} f_{ik} f_{jk}, \tag{6.6.9}$$

meaning that

$$\mathcal{L}^* u = -\sum_i \frac{\partial}{\partial x_i}(g_i u) + \frac{1}{2} \sum_{ijk} \frac{\partial^2}{\partial x_i \partial x_j}(f_{ik} f_{jk} u).$$

In the Stratonovich case the formula takes the form

$$\mathcal{L}^* = -\sum_i \frac{\partial}{\partial x_i} g_i + \frac{1}{2} \sum_k \left(\sum_i \frac{\partial}{\partial x_i} f_{ik} \right)^2. \tag{6.6.10}$$

The stochastic analog of (6.6.3) is

$$\frac{\partial u}{\partial s}(s,x) + \mathcal{L}u(s,x) = 0, \tag{6.6.11}$$

while the stochastic analog of (14.1.3) is

$$\frac{\partial \rho}{\partial t}(t,y) = \mathcal{L}^*\rho(t,y). \tag{6.6.12}$$

The analog of (6.6.5) is this:

Theorem 6.6.1 *Suppose that the coefficients f, g are smooth, and $u(s,x)$ is a smooth solution to (6.6.11) such that the derivatives $\partial u/\partial s$, and $\partial^2 u/\partial x_i \partial x_j$ are bounded. Then,*

$$u(s,x) = \mathbf{E}\phi(x(T)), \tag{6.6.13}$$

where $s \leq T$, and $u(T,x) = \phi(x)$. Conversely, if $\phi(x)$ is a smooth bounded function, then the function u defined by (6.6.13) is a smooth solution to (6.6.11).

Proof follows the same basic pattern as that of (6.6.5). Put $u_t = u(t,x(t))$. Then, by the Ito formula,

$$du_t = \left(\frac{\partial u}{\partial t}(t,x(t)) + \mathcal{L}u(t,x(t))\right)dt + \left\langle \frac{\partial u}{\partial x}, f(t,x(t))dw(t)\right\rangle, \tag{6.6.14}$$

which is equal to $\left\langle \frac{\partial u}{\partial x}, f(t,x(t))dw(t)\right\rangle$. Here, $\langle , \rangle$ stands for the Euclid scalar product. Integrating first over $[s,T]$, and then taking the mathematical expectations, we get that $u_s = \mathbf{E}u_T$, because the stochastic integral

$$\int_s^T \left\langle \frac{\partial u}{\partial x}, f(t,x(t))dw(t)\right\rangle$$

has vanishing mathematical expectation. In order to prove (6.6.13) it remains to note, that $u_s = u(s,x)$ and $u_T = \phi(x(T))$. Conversely, if $\phi(x)$ is a smooth bounded function, and the function $u = u(s,x)$ is defined by (6.6.13), then it is a smooth function (prove this, by using the preceding lecture!) with the following extra property: Suppose that ξ is (an arbitrary) random variable measurable w.r.t. the σ-algebra $\mathcal{F}_t = \sigma(w(\tau))$, where $\tau \leq t$. Then, $\xi u_t = \mathbf{E}(\xi u_T|\mathcal{F}_t)$ (explain why!), and

$$\mathbf{E}(\xi du_t) = 0 = \mathbf{E}\left(\xi\left(\frac{\partial u}{\partial t}(t,x(t)) + \mathcal{L}u(t,x(t))\right)dt\right),$$

which implies that $\frac{\partial u}{\partial t}(t, x(t)) + \mathcal{L}u(t, x(t)) \equiv 0$, because ξ is arbitrary. If we put $t = s$ now, we arrive at the desired conclusion that u is a smooth solution to (6.6.11). ▶

Now we turn to the differential equation satisfied by the distributional density of the random vector $x(t)$. This is a Schwartz distribution $p(t, y; s, x)$ with two groups of arguments: (s, x) is initial time and position, and (t, y) is final time and position. Our previous theorem says about differential equations satisfied by p with respect to the second group of variables. Our next theorem is about p considered as a function of the first group of variables. In other words, the distribution p is defined by $\int p(t, y)u(y)dy = \mathbf{E}u(x(t))$, where u is a test function (smooth with compact support).

Theorem 6.6.2 *The (generalized) function $p(t, y) = p(t, y; s, x)$ is a solution to (6.6.12) in the region $\{t \geq s\}$ such that $p(s, y) = \delta(x - y)$.*

Proof. Indeed, fix a test function u, and consider $\int p(t, y)u(y)dy = \mathbf{E}u(x(t))$. We have, first

$$\left(\int \frac{\partial}{\partial t} p(t, y)u(y)dy \right) dt = d\mathbf{E}u(x(t)) = \mathbf{E}du(x(t)). \tag{6.6.15}$$

Second, by the Ito formula, the right-hand side is

$$\mathbf{E}\mathcal{L}u(x(t)) = \int p(t, y)\mathcal{L}u(y)dy,$$

and third, $\int p(t, y)\mathcal{L}u(y)dy = \int \mathcal{L}^* p(t, y)u(y)dy$. Summing up, we have

$$\int \frac{\partial}{\partial t} p(t, y)u(y)dy = \int \mathcal{L}^* p(t, y)u(y)dy$$

for any test function u. This means that (6.6.12) holds for p. ▶

An interesting question is the conversion of the above Kolmogorov theorem. I do not know a completely satisfactory answer. The following results will do in the non-degenerate diffusion case.

Theorem 6.6.3 *Suppose, that the diffusion coefficient ff^* is non-degenerate, meaning that $\sum f_{ik} f_{jk} \xi_i \xi_j \geq C|\xi|^2$ for any $\xi \in \mathbf{R}^n$. Then, the* minimal *solution $u(t, y)$ to (6.6.12) in the region $\{t \geq s\}$ such that $u(s, y) = \delta(x - y)$ coincides with $p(t, y) = p(t, y; s, x)$.* ▶

The above theorems are characterizations of some simple stochastic functionals of a diffusion process in terms of Partial Differential Equations. Now we turn to a more complicated functional

$$\mathbf{E}\phi(x(t))e^{\int_s^t V(\tau,x(\tau))d\tau},$$

which is a continuous linear functional of the test function ϕ, and, thus, can be written as

$$\int \rho(t,y)\phi(y)dy = \mathbf{E}\phi(x(t))e^{\int_s^t V(\tau,x(\tau))d\tau}. \tag{6.6.16}$$

We are looking for the governing PDE for the (generalized) function ρ. The answer is given by the beautiful and important Feynman–Kac formula.

Theorem 6.6.4 *Assume (for simplicity) that the* potential V *is a bounded continuous function. Then,*

$$\frac{\partial \rho}{\partial t}(t,y) = (\mathcal{L}^* + V)\,\rho(t,y), \tag{6.6.17}$$

where $\mathcal{L}^*$ is the adjoint of the infinitesimal operator $\mathcal{L}$.

Proof does not differ essentially from that of Theorem 6.6.2 and is left as an exercise to the listener. ▶

In fact, for our main applications to the Kalman filtering we need to study the case, when V is not at all bounded. More specifically, the functional to be studied is

$$\int \rho(t,y)\phi(y)dy = \mathbf{E}\phi(x(t))e^{-\frac{1}{2}\int_s^t |h(\tau)|^2 d\tau + \int_0^t \langle h(\tau), dW(\tau)\rangle},$$

where W is a Wiener process independent of the Wiener process w related to the process x, while $h(\tau)$ is a shortening to $h(\tau, x(\tau))$. In this case, the governing PDE for $\rho(t,y)$ is a stochastic PDE first derived by M. Zakai. Namely, it is

$$d\rho(t,y) = \mathcal{L}^*\rho(t,y)dt + \rho(t,y)\langle h(t,y), dW(t)\rangle \tag{6.6.18}$$

in the Ito form, and

$$d\rho(t,y) = \left(\mathcal{L}^* - \frac{1}{2}|h|^2\right)\rho(t,y)dt + \rho(t,y)\langle h(t,y), dW(t)\rangle \tag{6.6.19}$$

in the Stratonovich form. Note, that (6.6.19) is formally identical to (6.6.17), while (6.6.18) is not.

Chapter 7

Pole placement

7.1 The Cauchy formula

We discuss another incarnation of the complete controllability for linear systems: the pole placement. The pole placement problem is as follows: Suppose we are given a linear control system

$$\dot{x} = Ax + Bu, \quad u \in U. \tag{7.1.1}$$

in the phase space $V = \mathbf{R}^n$. Then, if we define a linear feedback by

$$u = Cx \tag{7.1.2}$$

then we arrive at a linear dynamic system

$$\dot{x} = (A + BC)x, \tag{7.1.3}$$

and we might choose feedback so that the resulting dynamic system has desired properties. If the desired property is that the spectrum of $(A+BC)$ is a given set, one talks of the pole placement. Here, the pole can be understood as a pole of the resolvent of the matrix $A + BC$.

In fact, it is better to understand the spectrum of a matrix as a (positive) divisor of points (in the complex plane $\mathbf{C}$) with multiplicities, than just as a set. Another way to specify the spectrum of a matrix D is to specify its *characteristic polynomial* $p_D(s) = \det(s-D)$. This is a real monic polynomial (i.e., with the coefficient 1 for higher power of s) of degree equal to the size of the matrix D.

We say that the system (7.1.1) allows for an arbitrary pole placement if for any given monic polynomial p of degree $n = \dim V$, one can find a linear feedback $u = Cx$, so that $p_{A+BC} = p$.

Theorem 7.1.1 *System (7.1.1) allows for an arbitrary pole placement iff it is controllable.*

We prove first the necessity of controllability.

Suppose, that the system (7.1.1) is not controllable. Then, there exist a proper subspace $\mathcal{D} \subset V$ which is A-invariant and contains $W = BU$. The spectrum of $A + BC$ is the union of that of $A + BC$ in $\mathcal{D}$ and in the factor-space $\widehat{\mathcal{D}} = V/\mathcal{D}$. We notice, that the action of $A + BC$ in $\widehat{\mathcal{D}}$ does not depend on C, and coincides with that of A (prove this!). Therefore, part of the spectrum of $A + BC$ does not depend on feedback, and cannot be made arbitrary.

To prove the converse statement we reduce our task to the scalar completely controllable case. The reduction follows immediately (prove it!) from the general lemma [6]:

Lemma 7.1.2 *If the system (7.1.1) is controllable, then there exists a vector $b \in W = BU$, and a feedback matrix C such that the system*

$$\dot{x} = (A + BC)x + bv,\ v \in \mathbf{R} \tag{7.1.4}$$

with a scalar control, is controllable as well.

Proof is left to the listener as an exercise. ▲
(Hint: Take an arbitrary $0 \neq b \in W = BU$ and consider maximal d such that $\dim[b, Ab, \ldots, A^{d-1}b] = d$. If $d = \dim V$, we are done. Otherwise, put $Cb = CAb = \ldots = CA^{d-2}b = 0$, and $CA^{d-1}b = u'$, where $b' \stackrel{\text{def}}{=} Bu'$ is not contained in $[b, Ab, \ldots, A^{d-1}b]$. This is possible because of controllability, and if we substitute $A' = A + BC$ for A, the value of d will increase.)

Now, let us look more closely at the scalar control case.

In the general case,

$$\begin{aligned} \det(s - (A + BC)) &= \det\left((s - A)(1 - (s - A)^{-1}BC)\right) \\ &= \det(s - A)\det(1 - C(s - A)^{-1}B), \end{aligned} \tag{7.1.5}$$

where we used the identity $\det(1_n - \alpha\beta) = \det(1_m - \beta\alpha)$ valid for any pair α, β of $n \times m, m \times n$ matrices (prove it! Hint: Consider

$$\det \begin{pmatrix} 1_m & \beta \\ \alpha & 1_n \end{pmatrix}$$

and prove that it is equal to both $\det(1_n - \alpha\beta)$ and $\det(1_m - \beta\alpha)$). In the scalar case $\det(1-C(s-A)^{-1}B) = 1-C(s-A)^{-1}B$, where the 1×1 matrix in the right-hand side is identified with a scalar (and B, resp. C with column-, resp. row-vector).

Therefore, in the scalar case

$$\det(s - (A + BC)) = \det(s - A) - CF(s, A)B, \tag{7.1.6}$$

where $F(s, A) = [\det(s - A)](s - A)^{-1}$. (Note, that $CF(s, A)B$ is a scalar!) We notice that in the scalar case the pole placement is a *linear* (w.r.t. C) problem.

Denote the expression $CF(s, A)B$ by $\phi_C(s)$. We need to prove that the linear map $C \mapsto \phi_C$ is onto polynomials of degree $\leq (n-1)$. This is equivalent to the statement that $C \mapsto \phi_C$ has no kernel (why?). To prove this we observe that for an arbitrary polynomial (or analytic function) f one has

$$Cf(A)B = C\frac{1}{2\pi i}\int_\Gamma (s - A)^{-1} f(s)dsB = \frac{1}{2\pi i}\int_\Gamma \frac{\phi_C(s)f(s)ds}{\det(s - A)}, \tag{7.1.7}$$

where the contour Γ surrounds the spectrum of A. If $\phi_C(s) \equiv 0$, and f is any polynomial, then, by virtue of (7.3.3), $Cf(A)B = 0$. However, the vector $f(A)B$ for a suitable choice of f can be arbitrary, because of controllability. Thus, if $\phi_C(s) \equiv 0$, then $C = 0$.

Thus, ϕ_C can be an arbitrary polynomial of degree $n - 1$, and, in view of (7.1.6), $\det(s - (A + BC))$ can be an arbitrary monic polynomial of degree n. ▲

This is a polynomial matrix function of s, A of the form

$$F(s, A) = \sum_{j=0}^{n-1} s^j \phi_{n-1-j}(A), \tag{7.1.8}$$

where ϕ_j is a monic polynomial of degree j. Therefore, because of controllability, $e_j = \phi_j(A)B$, $j \in [0, n - 1]$ is a basis in the phase space $V = \mathbf{R}^n$. One can, therefore, choose the feedback C so that Ce_k is any given sequence of reals a_k, $k = 0, \ldots, n - 1$. This means that $CF(s, A)B$ can be an arbitrary polynomial of degree $n - 1$, and, in view of (7.1.6), $\det(s - (A + BC))$ can be an arbitrary monic polynomial of degree n.

We prove the converse statement in the following simplified form: Let Λ be a subset of the complex plane $\mathbf{C}$ such that $\#\Lambda = n$, and $A-\lambda$ is invertible

for any $\lambda \in \Lambda$. Suppose, that the system (7.1.1) is controllable. Then, there exists a feedback matrix C such that the spectrum of $A + BC$ coincides with Λ.

Consider the subsets $W_\lambda = (A-\lambda)^{-1}W$, where $\lambda \in \Lambda$. We want to choose a basis $\xi_\lambda \in W_\lambda$ of the phase space $V = \mathbf{R}^n$. To show that this is possible we have to prove that

1. The sum of W_λ coincides with V.

2. There is no linear relations of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \xi_\lambda = 0, \tag{7.1.9}$$

where $a_\lambda \in \mathbf{C}$ between $\xi_\lambda \in W_\lambda$

To prove 1 it suffices to show that the sum of W_λ contains W and is A-invariant. It contains W because of the identity

$$1 = \sum b_\lambda (A - \lambda)^{-1} \tag{7.1.10}$$

for some scalar b_λ. It is A-invariant because $A(A-\lambda)^{-1}w = w+(A-\lambda)^{-1}\lambda w$.

The equation (7.1.9) is impossible, because it implies

$$\sum_{\lambda \in \Lambda} a_\lambda (A - \lambda)^{-1} w = 0, \tag{7.1.11}$$

for any $w \in W$, and, therefore, for any $w \in V$.

Thus, we can choose a basis $\xi_\lambda \in W_\lambda$ of the phase space $V = \mathbf{R}^n$. For each element ξ_λ of the basis we have

$$(A - \lambda)\xi_\lambda = B\eta_\lambda, \tag{7.1.12}$$

and we put $C\xi_\lambda = -\eta_\lambda$. Therefore, $(A+BC)\xi_\lambda = \lambda\xi_\lambda$, and the set Λ coincides with the spectrum of $A + BC$.

7.2 Remarks on the Cauchy formula

7.2.1 Remark 1

An easier way of proving the Cauchy formula is given below suggested by a remark of Yura Taraban'ko.

Cauchy formula: Suppose f is a polynomial, A is $n \times n$ matrix, $\Omega \subset \mathbf{C}$ is a simply-connected domain in complex numbers, $\Gamma = \partial\Omega$ is the boundary of Ω oriented counter-clockwise. Suppose that Ω contains the spectrum of A (the set of eigenvalues). Then

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma}(s-A)^{-1}f(s)ds \tag{7.2.1}$$

Proof. Suppose $f(s) = \sum_{k\geq 0} f_k s^k$. One can replace Γ with a large circle, and rewrite

$$\widehat{f}(A) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\Gamma}(s-A)^{-1}f(s)ds$$

in the form

$$\frac{1}{2\pi i}\int_{\Gamma}\sum_{k\geq 0}\frac{A^k}{s^k}f(s)\frac{ds}{s} = \sum_{k\geq 0}A^k\frac{1}{2\pi i}\int_{\Gamma}s^{-k}f(s)\frac{ds}{s} \tag{7.2.2}$$

The integral $\frac{1}{2\pi i}\int_{\Gamma}s^{-k}f(s)\frac{ds}{s} = f_k$ by the residue calculus. Therefore,

$$\widehat{f}(A) = \sum_{k\geq 0}A^k f_k = f(A)$$

▲

One can derive from the Cauchy formula the Jordan (spectral) decomposition A in the following form

$$1 = \sum \pi_\lambda$$

$$A = \sum A_\lambda,$$

where λ runs over the spectrum of A, π_λ (the spectral projectors) are such that $\pi_\lambda^2 = \pi_\lambda$, $\pi_\lambda\pi_\mu = 0$, if $\lambda \neq \mu$, $A\pi_\lambda = \pi_\lambda A$, and $A_\lambda \stackrel{\text{def}}{=} \pi_\lambda A\pi_\lambda$. Moreover, λ is the only eigenvalue of A_λ, and therefore $(A_\lambda - \lambda)^N = 0$, if N is sufficiently large. The projector π_λ can be defined via

$$\pi_\lambda = \frac{1}{2\pi i}\int_{\Gamma(\lambda)}(s-A)^{-1}ds,$$

where $\Gamma(\lambda)$ is a small contour around λ.

Optional Exercise. Prove all the above statements about spectral decomposition.

7.2.2 Remark 2

One has to define A_λ as the operator $A : V \to V$ restricted to the invariant space $V_\lambda \stackrel{\text{def}}{=} \pi_\lambda V$. So, the new and old A_λ coincide in their common domain V_λ (but the old A_λ has a greater domain V). The spectrum of old A_λ might contain 0, besides of λ.

Thus, the correct statement of the spectral decomposition is this:

$$1 = \sum \pi_\lambda$$

$$A = \sum \pi_\lambda A_\lambda \pi_\lambda,$$

where λ runs over the spectrum of A, π_λ (the spectral projectors) are such that $\pi_\lambda^2 = \pi_\lambda$, $\pi_\lambda \pi_\mu = 0$, if $\lambda \neq \mu$, $A\pi_\lambda = \pi_\lambda A$, and $A_\lambda : V_\lambda \to V_\lambda$ is the operator $A : V \to V$ restricted to the invariant space $V_\lambda \stackrel{\text{def}}{=} \pi_\lambda V$. Moreover, λ is the only eigenvalue of A_λ.

You have already checked that the integral formula

$$\pi_\lambda = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A)^{-1} ds \tag{7.2.3}$$

defines the spectral projectors π_λ with required properties, so that, in particular, $V_\lambda \stackrel{\text{def}}{=} \pi_\lambda V$ is an invariant subspace.

Therefore

$$A_\lambda = \pi_\lambda A \pi_\lambda = A\pi_\lambda = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A)^{-1} A ds = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A)^{-1} s ds, \tag{7.2.4}$$

where all operators are considered as defined on $V_\lambda \stackrel{\text{def}}{=} \pi_\lambda V$. The latter identity holds because $\int_{\Gamma(\lambda)} ds = 0$. Note also that the integral

$$\frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A)^{-1} s ds = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A_\lambda)^{-1} s ds$$

since $A = A_\lambda$ on V_λ. Now, we have to show that if $\mu \neq \lambda$, then $(A_\lambda - \mu)$ is invertible. We define the putative $(A_\lambda - \mu)^{-1}$ by the integral formula

$$S = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (s - A_\lambda)^{-1} \frac{ds}{s - \mu} \tag{7.2.5}$$

and check that $S(A_\lambda - \mu) = 1$ (of course, as operators on V_λ).

Indeed, by the standard trick with the resolvent identity

$$S(A_\lambda - \mu) = \frac{1}{2\pi i}\int_{\Gamma(\lambda)} (s - A_\lambda)^{-1} \left(\frac{1}{2\pi i}\int_{\Gamma'(\lambda)} \frac{(t-\mu)dt}{t-s} \right) \frac{ds}{s-\mu} = 1, \quad (7.2.6)$$

since the inner integral is $s - \mu$, and because of the integral formula for $\pi_\lambda = 1$ on V_λ.

Perhaps, a better way is to extend the Cauchy formula from polynomials to analytic functions in the neighborhood of λ, including the function $s \mapsto \frac{1}{s-\mu}$.

7.3 Exercises

We are given a linear control system

$$\dot{x} = Ax + Bu, \quad u \in U. \quad (7.3.1)$$

in the phase space $V = \mathbf{R}^n$. Characteristic polynomial of a matrix D is defined by $p_D(s) = \det(s - D)$.

Exercise 1. Give a detailed proof of the following lemma [6]:

Lemma 7.3.1 *If the system (7.3.1) is controllable, then there exists a vector $b \in W = BU$, and a feedback matrix C such that the system*

$$\dot{x} = (A + BC)x + bv,\ v \in \mathbf{R} \quad (7.3.2)$$

with a scalar control, is controllable as well.

(Hint: Take an arbitrary $0 \neq b \in W = BU$ and consider maximal d such that $\dim[b, Ab, \ldots, A^{d-1}b] = d$. If $d = \dim V$, we are done. Otherwise, put $Cb = CAb = \ldots = CA^{d-2}b = 0$, and $CA^{d-1}b = u'$, where $b' \stackrel{\text{def}}{=} Bu'$ is not contained in $[b, Ab, \ldots, A^{d-1}b]$. This is possible because of controllability, and if we substitute $A' = A + BC$ for A, the value of d will increase.)

Exercise 2. Prove the identity $\det(1_n - \alpha\beta) = \det(1_m - \beta\alpha)$ for any pair α, β of $n \times m, m \times n$ matrices (Hint: Consider

$$\det \begin{pmatrix} 1_m & \beta \\ \alpha & 1_n \end{pmatrix}$$

and prove that it is equal to both $\det(1_n - \alpha\beta)$ and $\det(1_m - \beta\alpha)$)

Exercise 3. Prove that the matrix $F(s,A) \stackrel{\text{def}}{=} [\det(s-A)](s-A)^{-1}$ has polynomials of s of degree less or equal to $n-1$ as its matrix elements. (Hint: use explicit formulas for these matrix elements.)
Exercise 4. Prove the Cauchy formula: Suppose f is a polynomial, A is $n \times n$ matrix, $\Omega \subset \mathbf{C}$ is a domain in complex numbers, $\Gamma = \partial\Omega$ is the boundary of Ω oriented counter-clockwise. Suppose that Ω contains the spectrum of A (the set of eigenvalues). Then

$$f(A) = \frac{1}{2\pi i}\int_\Gamma (s-A)^{-1} f(s) ds \tag{7.3.3}$$

Hint 1: Define $\widehat{f}(A) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_\Gamma (s-A)^{-1} f(s) ds$, and show that it does not depend on Γ, and solve the following sequence of exercises:
Exercise 4a. Prove that, if $f \equiv 1$ then

$$\widehat{f}(A) = 1_n \tag{7.3.4}$$

(Hint: replace Γ with a large circle)
Exercise 4b. Prove that, if $f(s) = s$ then

$$\widehat{f}(A) = A \tag{7.3.5}$$

Exercise 4c. Prove that $\widehat{fg}(A) = \widehat{f}(A)\widehat{g}(A)$. (Hint: write

$$\widehat{f}(A)\widehat{g}(A) = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} (s-A)^{-1} f(s) ds (t-A)^{-1} g(t) dt \tag{7.3.6}$$

and use the resolvent identity

$$(s-A)^{-1}(t-A)^{-1} = ((s-A)^{-1} - (t-A)^{-1})\frac{1}{t-s}$$

to rewrite (7.3.6) as the difference of

$$\frac{1}{2\pi i}\int_\Gamma (s-A)^{-1} f(s) \left(\frac{1}{2\pi i}\int_{\Gamma'} \frac{g(t)}{t-s} dt\right) ds \tag{7.3.7}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma'} (t-A)^{-1} g(t) \left(\frac{1}{2\pi i}\int_{\Gamma} \frac{f(s)}{t-s} ds\right) dt \tag{7.3.8}$$

If Γ' is a very "large" contour compared to Γ, then the inner integral in (7.3.7) is equal to $g(s)$ while that in (7.3.8) is zero)
Hint 2: Show that (7.3.3) holds for diagonalizable matrices A, and get the general case by passing to the limit.
Hint 3: Take a large circle for the contour γ. This is possible by Hint 1. Then

$$(s - A)^{-1} = \sum_{n=0}^{\infty} A^n s^{-n-1} \tag{7.3.9}$$

on γ. Then,

$$\widehat{f}(A) = \sum_{n=0}^{\infty} A^n \frac{1}{2\pi i} \int_{\Gamma} s^{-n-1} f(s) ds. \tag{7.3.10}$$

Show that if $f(s) = \sum a_n s^n$, then $\frac{1}{2\pi i} \int_{\Gamma} s^{-n-1} f(s) ds = a_n$.

Chapter 8

Elements of Nonlinear Controllability

8.1 Lie brackets

First, we indicate an interpretation of the Kalman controllability criterion from the geometric point of view.

A general control system is given by a set of vector fields f_u in the phase space $V = \mathbf{R}^n$:

$$\dot{x} = f_u(x) \tag{8.1.1}$$

We assume that each field is infinitely smooth (w.r.t. x). The set of all infinitely smooth vector fields is more than just a linear $\mathbf{R}$-space (or even a module over C^∞ functions), it is a Lie algebra, and now we explain what this means.

Note first, that a vector field $f(x)$ can be regarded as a linear operator — the operator of differentiation along the field. In other words, f defines and is defined by the operator $\langle f, \frac{\partial}{\partial x}\rangle \stackrel{\text{def}}{=} \sum f_i \frac{\partial}{\partial x_i}$. The operator $\langle f, \frac{\partial}{\partial x}\rangle$ can be defined as follows:

$$\langle f, \frac{\partial}{\partial x}\rangle u(x) = \left.\frac{d}{dt}\right|_{t=0} u(\phi_t(x)),$$

where $u(x)$ is an arbitrary smooth function, and $\phi_t(x)$ is any smooth map such that $\phi_0(x) = x$ and $\left.\frac{d}{dt}\right|_{t=0} \phi_t = f$. In particular, if $e^{tf}(x)$ stands for the integral curve of f through x, then $\langle f, \frac{\partial}{\partial x}\rangle u(e^{tf}(x)) = \frac{d}{dt} u(e^{tf}(x))$. In what follows we'll often denote $\langle f, \frac{\partial}{\partial x}\rangle$ simply by f. Then, the latter equality

may be rewritten as $\frac{d}{dt}e^{tf} = fe^{tf}$, where both parts are now interpreted as operators acting upon smooth functions, the operator e^{tf} being defined as $e^{tf}u(x) = u(e^{tf}(x))$.

Upon interpreting vector fields as operators one can multiply them. Unfortunately, the vector fields are first order differential operators, and their product is a second order operator, — not a vector field. However, the commutator $[a,b] \stackrel{\text{def}}{=} ab - ba$ of two first order differential operators is again a first order differential operator, i.e., a vector field.

Exercise 8.1.1 *Check this.*

The commutator determines a structure of the Lie algebra on the vector space of all smooth vector fields. This means that

1. Operation $f, g \mapsto [f,g]$ is bilinear.
2. $[f,f] = 0$ for any f
3. The Jacobi identity holds: $[[f,g],h] + [[g,h],f] + [[h,f],g] = 0$

Exercise 8.1.2 *Check the Jacobi identity.*

Informally speaking, any Lie algebra is naturally identified with the tangent space in the neutral element (unit) of a "Lie group". In the case of smooth vector fields this group is the group of diffeomorphisms (invertible smooth maps). This is actually a corollary of the above formula $\frac{d}{dt}e^{tf} = fe^{tf}$.

Any set $\mathcal{F}$ of vector fields f_u generate a Lie algebra $Lie(\mathcal{F}) = Lie(f_u)$, which is a minimal Lie subalgebra (in all vector fields) containing all f_u. As a vector space it is generated by multiple commutators like $[\ldots[[f_{u_1}, f_{u_2}], \ldots, f_{u_n}]$. For any $x \in V$ we denote by $\mathcal{F}(x)$ the vector space consisting of values $g(x)$ in x of all vector fields $g \in \mathcal{F}$. In particular, $Lie(\mathcal{F})(x)$ consists of vectors $g(x)$ for all $g \in Lie(\mathcal{F})$.

Exercise 8.1.3 *Check that* $[f,g] = \langle f, \frac{\partial}{\partial x}\rangle g - \langle g, \frac{\partial}{\partial x}\rangle f$, *where* f, g *in the right-hand side are interpreted as smooth vector-valued functions.*

Exercise 8.1.4 *Prove that* $e^{tf}e^{tg}e^{-tf}e^{-tg}(x) = x + t^2[f,g] + o(t^2)$. *Make picture.*

Exercise 8.1.5 *Find a connection between Poisson and Lie brackets. More precisely, to a smooth Hamiltonian $H(p,q)$ one associates the vector field $\xi_H = (-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p})$. Calculate $[\xi_{H_1}, \xi_{H_2}]$.*

Kalman criterion

Now we turn to the vector fields in the Kalman setup:

$$\dot{x} = Ax + Bu, \quad u \in U = \mathbf{R}^m, \tag{8.1.2}$$

One can easily check that $[Ax, a] = -Aa$, where a is an arbitrary constant vector field (vector). Therefore, the Lie algebra $Lie(\mathcal{F})$ generated by the Kalman fields $f_u = Ax + Bu$ can be described as follows: As a vector space it is spanned by the linear field Ax and the constant fields $Bu, ABu, \ldots, A^{n-1}Bu$.

Exercise 8.1.6 *Prove the above statement*

We see that the Kalman controllability condition can be restated as follows: For all $x \in V$, $Lie(\mathcal{F})(x)$ is maximal possible, i.e., coincides with the entire tangent space at x. Here, $\mathcal{F}$ is the set of the Kalman vector fields $f_u = Ax + Bu$.

Exercise 8.1.7 *Prove the above statement (hint: put $x = 0$)*

Definition. Any family $\mathcal{F}$ with this property $\dim Lie(\mathcal{F})(x) = n$ is called completely nonholonomic.

Therefore, the Kalman criterion can be stated as follows: The family $\mathcal{F} = \{f_u = Ax + Bu\}$ is completely nonholonomic. In the next section we discuss to what extent this restatement of the Kalman criterion is applicable to nonlinear control systems.

8.2 Rashevsky – Chow theorem

It is convenient to use piecewise constant controls when talking about nonlinear controllability.

Let $\mathcal{F}$ be any set of smooth vector fields such that if $f \in \mathcal{F}$, then the phase flow e^{tf} is defined for any t (complete vector field). The set $M(\mathcal{F})$ of diffeomorhisms of the form

$$M(\mathcal{F}) = \{e^{t_1 f_1} \ldots e^{t_n f_n}\}, \tag{8.2.1}$$

where $t_i \geq 0$, and $f_i \in \mathcal{F}$, is called the semigroup generated by $\mathcal{F}$. These are exactly the transformations generated by piecewise constant controls.

Similarly, the set $G(\mathcal{F})$ of diffeomorhisms of the form

$$G(\mathcal{F}) = \{e^{t_1 f_1} \dots e^{t_n f_n},\ t_i \in \mathbf{R},\ f_i \in \mathcal{F}\} \tag{8.2.2}$$

is called the group generated by $\mathcal{F}$.

Note, that the only difference between $M(\mathcal{F})$ and $G(\mathcal{F})$ is in the positivity condition $t_i \geq 0$ in the M-case, and $G(\mathcal{F}) = M(\mathcal{F} \bigcup -\mathcal{F})$.

Then, the attainable set $\mathcal{A}(x)$ from a point x is by definition

$$\mathcal{A}(x) = \mathcal{A}(\mathcal{F})(x) = \{mx,\ m \in M(\mathcal{F})\},$$

while the orbit $\mathcal{O}(x)$ of a point x is by definition

$$\mathcal{O}(x) = \mathcal{O}(\mathcal{F})(x) = \{gx,\ g \in G(\mathcal{F})\}.$$

Now the main result of the section is as follows:

Theorem 8.2.1 (Rashevsky – Chow) *Suppose that the family $\mathcal{F}$ is completely nonholonomic, i.e.,* $\dim Lie(\mathcal{F})(x) = n$ *for any point x. Then the orbit $\mathcal{O}(x)$ of any $x \in V$ coincides with the entire phase space. In other words, the system $\mathcal{F} \bigcup -\mathcal{F}$ is completely controllable by piecewise constant controls.*

Counterexample. Consider $V = \mathbf{R}^2$ with $\mathcal{F} = \{\frac{\partial}{\partial x_1}, a(x_1)\frac{\partial}{\partial x_2}\}$, where $a \not\equiv 0$ has a compact support. Then, $\mathcal{O}(\mathcal{F})(x) = V$ for any $x \in V$, but the system is not completely nonholonomic.

Exercise 8.2.2 *Prove the above statement.*

Theorem 8.2.3 (converse Rashevsky – Chow) *Suppose that the family $\mathcal{F}$ consists of analytic vector fields and the orbit $\mathcal{O}(x)$ of a point $x \in V$ coincides with the entire phase space. Then the system $\mathcal{F}$ is completely nonholonomic.*

To prove the Rashevsky – Chow theorem we need some extra machinery. First, define the action of a diffeomorphism $F : V \to V$ upon vector field f by the formula $F_* f(x) = \left(\frac{\partial F}{\partial x}\right) f(F^{-1}x)$. In other words, we take the tangent vector $f(F^{-1}x)$ at the point $F^{-1}x$ and transport it to the point x by the tangent map $\frac{\partial F}{\partial x}$ of the diffeomorphism F. This action is denoted also by $\operatorname{Ad} F$. If one treats F and f as operators on smooth functions, one has $\operatorname{Ad} F(f) = FfF^{-1}$. Another important operator $\operatorname{ad} f$ on vector fields

is associated to any vector field f, and is defined via $\operatorname{ad} f(g) = [f, g]$. The above operations are related via $\frac{d}{dt} \operatorname{Ad} e^{tf}(g) = \operatorname{Ad} e^{tf} \operatorname{ad} f(g)$.

For any set $\mathcal{P}$ of diffeomorphisms and a family $\mathcal{F}$ of vector fields we define a new family $\operatorname{Ad} \mathcal{P}(\mathcal{F}) = \{\operatorname{Ad} P(f),\ P \in \mathcal{P},\ f \in \mathcal{F}\}$, and define a linear hull $[\mathcal{F}]$ of any family $\mathcal{F}$ as the vector space generated by $\mathcal{F}$. Then we have the following simple statement, which is the core of the Rashevsky – Chow theorem(s).

Proposition 8.2.4 *Suppose, we are given a family $\mathcal{F}$ of smooth complete vector fields. Put $G = G(\mathcal{F})$ which is the group generated by $\mathcal{F}$. Then,*

$$Lie(\mathcal{F})(x) \subset [\operatorname{Ad} G(\mathcal{F})](x) \tag{8.2.3}$$

for any point x. If the family $\mathcal{F}$ consists of analytic vector fields, the above inclusion turns into equality

$$Lie(\mathcal{F})(x) = [\operatorname{Ad} G(\mathcal{F})](x). \tag{8.2.4}$$

Proof. Upon a brief reflection (give a detailed account!) the inclusion (8.2.3) reduces to

$$\operatorname{ad} f(g)(x) = [f, g](x) \in [\operatorname{Ad} e^{tf}(g)](x), \tag{8.2.5}$$

for any couple f, g of smooth vector fields. This is almost obvious, since

$$\operatorname{ad} f(g)(x) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad} e^{tf}(g)(x) \in [\operatorname{Ad} e^{tf}(g)](x), \tag{8.2.6}$$

because the curve $t \mapsto \operatorname{Ad} e^{tf}(g)$ is contained in $\{\operatorname{Ad} e^{tf}(g)\}$, and its tangent vector $\frac{d}{dt} \operatorname{Ad} e^{tf}(g)$ is contained in the linear hull $[\operatorname{Ad} e^{tf}(g)]$. Similarly, to obtain (8.2.4) we notice that the curve $t \to \operatorname{Ad} e^{tf}(g)(x)$ is analytic if f, g are analytic vector fields, and, therefore, its linear span coincides with that of the coefficients of its Taylor series at $t = 0$. These coefficients coincide up to a scalar multiple with vectors of the form $(\operatorname{ad} f)^n g(x)$ which belong to $Lie(f, g)(x)$. ▲

Now we can prove the (direct) Rashevsky – Chow theorem. Under its assumptions we have, by virtue of Proposition 8.2.4, that for any $x \in V$ $\dim[\operatorname{Ad} G(\mathcal{F})](x) = n$. In other words, one can chose n vector fields g_i of the form $g_i = \operatorname{Ad} G_i(f_i)$, where $G_i \in G(\mathcal{F})$ and $f_i \in \mathcal{F}$, such that $g_i(x)$ are linearly independent. It is clear (explain!) that $e^{tg_i} = \operatorname{Ad} G_i(e^{tf_i}) \in G(\mathcal{F})$. Therefore, the map $G : \mathbf{R}^n \to \mathcal{O}(x) \subset V$, given by $G(t_1, \dots, t_n) =$

$e^{t_1 g_1} \dots e^{t_n g_n}(x)$, has maximal possible rank n at $0 \in \mathbf{R}^n$. Now the implicit function theorem shows that $\mathcal{O}(x)$ contains an entire neighborhood of x. This implies (prove it!) that $\mathcal{O}(x)$ is an open subset of the phase space V. Since $V = \bigcup_{x \in V} \mathcal{O}(x)$ we get a decomposition of the connected space V into disjoint union (give detailed explanation!) of open sets. Therefore, $\mathcal{O}(x) = V$ for any $x \in V$ and the direct Rashevsky – Chow theorem is proved.

We will not touch the proof of the converse Rashevsky – Chow theorem, for a natural approach to it requires more prerequisites from the (elementary) geometry of manifolds. So, we leave the proof to the keen reader.

8.3 Hörmander theorem

This purely illustrative section is aimed at convincing the reader that non-holonomic families of vector fields are important well beyond the control theory. We present here a deep theorem of L. Hörmander about hypoelliptic differential operators of second order.

Definition. Let P be a linear differential operator with smooth ($C^\infty(V)$) coefficients in $V = \mathbf{R}^n$. P is said to be hypoelliptic if every local solution (i.e., solution in an open subset) u of the equation $Pu = f$ is smooth, provided that f is smooth.

It is classically well-known that, say, the Laplace operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ resp. the heat operator $\frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is hypoelliptic in $\mathbf{R}^n$ resp. $\mathbf{R}^{n+1}$. A.N. Kolmogorov have shown in 30-th that the operator $\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}$ is hypoelliptic in $\mathbf{R}^2$.

Theorem 8.3.1 (Hörmander) *Suppose we are given a finite completely non-holonomic family $X_0, X_1, \dots, X_l$ of smooth vector fields. Then the operator $P = X_0 + \sum_{i=1}^l X_i^2$ is hypoelliptic.*

Exercise 8.3.2 *Show that the Hörmander theorem implies hypoellipticity of the above three operators: Laplace, heat, Kolmogorov's.*

Chapter 9

Black box

Suppose we are given a time-invariant observable linear control system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad u \in U = \mathbf{R}^m, \\ y &= Cx, \end{aligned} \tag{9.1.1}$$

where the phase space is $V = \mathbf{R}^n$, and the space of observable vectors $W = \mathbf{R}^p$. If we assume the "initial condition" $x(0) = 0$ there arises the input-output map S given by

$$y(t) = (Su)(t) = \int_0^t Ce^{(t-s)A}Bu(s)ds \tag{9.1.2}$$

We will refer to (9.1.1) as system (A, B, C, V).

Now the main problem is: Is it possible to recover the entire system (9.1.1) if the input-output map S is known? (see [5])

Obvious answer is no, because of two reasons: 1) if the system (9.1.1) is not controllable, then one can replace the phase space V with a smaller space (subspace) which is A-invariant, and contains BU, and this replacement does not affect the input-output map, 2) similarly, if the system (9.1.1) is not observable, then one can replace V with a smaller factorspace $V/\mathcal{P}$ w.r.t. the subspace $\mathcal{P}$ of undetectable elements (i.e., $x \in V$ such that $t \mapsto Ce^{At}x \equiv 0$), and this again does not affect the input-output map.

Actually, if we are interested only in the input-output map, there is no need to consider (9.1.1) in case it is not both controllable and observable. Indeed, one can form canonically from (9.1.1) another controllable and observable system with the same input-output map. To do so, one just has to,

first, replace V with the attainable set from the origin, and, second, factor it out by the space of undetectable elements.

Now we can refine our initial question: Suppose that the system (9.1.1) is controllable and observable. Is it possible then to recover the system (9.1.1) from the input-output map?

In other words, we consider the controllable and observable system (9.1.1) as a content of a "black box" with known input-output.

It is natural to consider two systems (A', B', C', V') and (A'', B'', C'', V'') equivalent if there exists a linear isomorphism $D : V'' \to V'$ such that

$$A' = DA''D^{-1},\ B' = DB'',\ C'' = C'D. \tag{9.1.3}$$

Theorem 9.1.3 *Two controllable and observable systems*

$$(A', B', C', V') \text{ and } (A'', B'', C'', V'')$$

with the same input-output map (9.1.2) are equivalent.

Proof. Suppose, the system (A, B, C, V) is controllable and observable. We will build from the input-output map (9.1.2) a system $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{V})$ with the same input-output map, and will show that it is equivalent to the initial system.

First, the map (9.1.2) is a convolution with the kernel

$$\mathcal{K}(s) = \begin{cases} Ce^{As}B, & \text{if } s \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{9.1.4}$$

Thus, the knowledge of S is the same thing as the knowledge of the matrix function

$$\mathcal{F}(s) = Ce^{As}B. \tag{9.1.5}$$

We define the canonical phase space $\widetilde{V}$ as follows: for each U-valued polynomial $f(s) = \sum f_k s^k,\ f_k \in U$ we consider the function

$$\widehat{f}(s) = \sum \left(\frac{\partial}{\partial s}\right)^k \mathcal{F}(s) f_k = \sum CA^k e^{As} B f_k \tag{9.1.6}$$

The space $\widetilde{V}$ consists of these functions $\widehat{f}$. We define the map $\widetilde{B} : U \to \widetilde{V}$ as $\widetilde{B}u(s) = \mathcal{F}(s)u$, and the map $\widetilde{C} : \widetilde{V} \to W$ as $\widetilde{C}\widehat{f} = \widehat{f}(0)$. The map $\widetilde{A} : \widetilde{V} \to \widetilde{V}$ is defined as $\frac{\partial}{\partial s}$. We have to check that the system $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{V})$ is equivalent to (9.1.1).

We define the equivalence map $D : V \to \widetilde{V}$ as follows: it maps a vector $x \in V$ to the function $\widehat{f}(s) = Ce^{As}x$. The definition is correct, because, by virtue of controllability, x can be represented in the form $\sum A^k B f_k$ (why?), and, therefore, $\widehat{f}(s) = \sum \left(\frac{\partial}{\partial s}\right)^k \mathcal{F}(s) f_k$ belongs to $\widetilde{V}$. These arguments show also that the map D is onto, while the kernel of D is zero because of observability.

Thus, D is an isomorphism, and a direct verification confirms that it is an equivalence. Indeed,

$$\widetilde{A}Dx(s) = \frac{\partial}{\partial s} Ce^{As}x = Ce^{As}Ax = DAx(s),$$

$$\widetilde{B}Du(s) = Ce^{As}Bu = DBu(s),$$

and

$$\widetilde{C}Dx = Ce^{As}x|_{s=0} = Cx$$

▲

Exercise 9.1.4 *State and prove a similar theorem on uniqueness of a linear system (A, B, C, V) with a given transfer function $G : \lambda \mapsto C(\lambda - A)^{-1}B$.*

Hint: One can reduce the question to Theorem 9.1.3 by considering the Fourier-Laplace transform of the transfer function. Indeed, the function

$$\Phi(s) = \begin{cases} e^{As} & , \quad \text{if } s > 0 \\ 0 & , \quad \text{if } s \leq 0 \end{cases} \tag{9.1.7}$$

is a solution of

$$\left(\frac{\partial}{\partial s} - A\right) \Phi = \delta, \tag{9.1.8}$$

so that the Fourier-Laplace transform $\widehat{\Phi}$ solves $(\lambda - A)\widehat{\Phi}(\lambda) = 1$, and, therefore, G is the Fourier-Laplace transform of the function (9.1.4). Another way of reasoning is this:

Theorem 9.1.5 *Two controllable and observable systems*

$$(A', B', C', V') \text{ and } (A'', B'', C'', V'')$$

with the same transfer function $G : \lambda \mapsto C(\lambda - A)^{-1}B$ are equivalent.

Proof follows closely that of Theorem 9.1.3. The problem is to build from the transfer function $G(\lambda) = C(\lambda - A)^{-1}B$ a system $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{V})$ which is equivalent to (A, B, C, V).

First, we note that

$$Cf(A)B = \frac{1}{2\pi i}\int_\Gamma G(s)f(s)ds \tag{9.1.9}$$

for any polynomial f, where Γ is any sufficiently large circle and denote the matrix in the right-hand side of (9.1.9) by $\widehat{f}$. This follows immediately from the Cauchy formula of lecture 6.

Define a vector space $\widetilde{\widetilde{V}}$ as follows: It consists of U-valued functions of the form $s \mapsto \sum f_k(s)u_k$, where the sum is finite, f_k is a (scalar) polynomial, and $u_k \in U$.

We notice that $\widetilde{\widetilde{V}}$ is a module over the polynomials $\mathbf{R}[s]$, meaning that if $v(s)$ is a polynomial, and $f = \sum f_k(s)u_k \in \widetilde{\widetilde{V}}$ the product

$$vf = \sum v(s)f_k(s)u_k \in \widetilde{\widetilde{V}}$$

is well defined. In particular, operator $\widetilde{\widetilde{A}}$ is defined correctly by

$$\widetilde{\widetilde{A}}\sum f_k(s)u_k = s\sum f_k(s)u_k = \sum sf_k(s)u_k.$$

We can also define naturally an operator $\widetilde{\widetilde{B}} : U \to \widetilde{\widetilde{V}}$ by $\widetilde{\widetilde{B}}(u) = u$.

Another important operator $\widetilde{\widetilde{C}} : \widetilde{\widetilde{V}} \to W$ is defined via

$$\widetilde{\widetilde{C}}\left(\sum f_k(s)u_k\right) = \sum \widehat{f_k}u_k,$$

where the notation $\widehat{f_k}$ is defined after formula (9.1.9). Then define the subspace $I \subset \widetilde{\widetilde{V}}$ as follows: a vector $f = \sum f_k(s)u_k \in \widetilde{\widetilde{V}}$ is contained in I iff

$$\sum \widehat{(vf_k)}u_k = 0$$

for any polynomial v. In other words, $f \in I$ iff $\widetilde{\widetilde{C}}\widetilde{\widetilde{A}}^k f = 0$ for any $k = 0, 1, \ldots$ It is clear, by the way, that $I \subset \widetilde{\widetilde{V}}$ is a submodule of $\widetilde{\widetilde{V}}$ which means that if $f \in I$, then $vf \in I$ for any (scalar) polynomial v.

Now we put $\widetilde{V} = \widetilde{\widetilde{V}}/I$, and similarly factor the operators $\widetilde{A}$, $\widetilde{\widetilde{B}}$, and $\widetilde{\widetilde{C}}$. It remains to define the equivalence $D : \widetilde{V} \to V$ by

$$D\left(\sum f_k(s)u_k\right) = \sum f_k(A)Bu_k.$$

Here, $\sum f_k(s)u_k$ is a representative in $\widetilde{\widetilde{V}}$ of an element of $\widetilde{V} = \widetilde{\widetilde{V}}/I$. The definition is correct, since if $f = \sum f_k(s)u_k \in I$, then $CA^kDf = 0$ for any k, and, therefore, by virtue of observability, $Df = 0$.

Another natural question is: How to characterize the possible input-output maps associated with linear systems via (9.1.2)? In other words, what characterizes the matrix functions of the form $Ce^{As}B$?

Exercise 9.1.6 *A matrix function $\mathcal{F}(s) : U \to W$ has the form $Ce^{As}B$ iff the vector space generated by $\left(\frac{\partial}{\partial s}\right)^k \mathcal{F}(s)$, $k \geq 0$ is finite dimensional. This is also equivalent to the statement that $\mathcal{F}(s) = \sum a_{m\lambda}s^m e^{\lambda s}$ is a quasipolynomial.*

Hint: Follow the construction in the proof of the previous theorem.

Exercise 9.1.7 *State and prove a similar theorem on characterization of the transfer function $C(\lambda - A)^{-1}B$ of a linear system (A, B, C, V).*

Hint: $G(\lambda) = C(\lambda - A)^{-1}B$ is a rational function of λ such that $G(\lambda) \to 0$ as $\lambda \to \infty$. Here, rational means that every matrix element is rational. To establish the converse follow the proof of Theorem 9.1.5.

Chapter 10

Examples

10.1 A cart under bounded force

We will discuss two simple examples of linear control systems with bounded control with emphasis on explicit presentation of the corresponding attainable sets.

The first system is

$$\ddot{x} = u,\ |u| \le 1, \tag{10.1.1}$$

which is a particular case of

$$x^{(n)} = u,\ |u| \le 1. \tag{10.1.2}$$

We can immediately write down the support function $H_{D(T)}(\xi)$ of the attainable set $D(T)$ (from the origin) at time T. We rewrite (10.1.2) in the form of the 1st order system by using coordinates $x_i = x^{(i-1)}$ for $i = 1, \dots, n$. Then, the general formula $H_{D(T)}(\xi) = \int_0^T |B^* e^{A^* t}\xi|\, dt$ takes the form

$$H_{D(T)}(\xi) = \int_0^T \left| \sum_{i=1}^{n} \xi_i \frac{t^{n-i}}{(n-i)!} \right| dt \tag{10.1.3}$$

The remarkable fact about this support function and, therefore, about the corresponding attainable set is that the function and set do not depend on time T essentially, meaning that everything is constant in coordinates $X_i = x_i/T^{n-i+1}$. In other words, if the matrix $A = A(T)$ is given by $(Ax)_i = x_i/T^{n-i+1}$, then $A(T)D(T)$ does not depend on time. Similarly, the function

$H_{D(T)}(A(T)^*\xi)$ does not depend on time. We write $A(T)^*$ instead of $A(T)$ "for functoriality", although $A(T)^* = A(T)$. So it suffices to study $D = D(1)$

The set D has interesting singularities of its boundary. The gradient map $x(\xi) = \frac{\partial H}{\partial \xi}$ from the unit sphere $|\xi| = 1$ to the boundary ∂D of D is given by

$$x(\xi)_k = \int_0^1 \text{sign} \left(\sum_{i=1}^{n} \xi_i \frac{t^{n-i}}{(n-i)!} \right) \frac{t^{n-k}}{(n-k)!} dt, \tag{10.1.4}$$

and the multiple points of this map corresponds to singularities of the boundary. One can identify the entire phase space with polynomials

$$x = \sum_{0}^{n-1} x_i t^{n-i}$$

of degree $\leq (n-1)$. Then multiple points corresponds to polynomials that have less than $(n-1)$ odd zeroes in the interval $[0,1]$. These are exactly the singular points of the boundary. The regular points corresponds to polynomials such that all their roots are simple and are contained in $[0,1]$.

Exercise 10.1.1 *Prove the above statements for $n = 2$. Make a picture of the attainable set.*

Theorem 10.1.2 *Suppose, we are given a linear control system*

$$\dot{x} = Ax + Bu, u \in U, \tag{10.1.5}$$

where U is a strictly convex compact, and suppose that $x \in \partial D(T)$ is a point of the boundary of the attainable set (from zero), corresponding to time T. Then, the control $u(t)$, $t \in [0,T]$, which brings zero to x, is defined uniquely.

Proof of this important result is left to the reader as an exercise.

10.2 Harmonic Oscillator

Here, we consider the system

$$\ddot{x} + x = u, \ |u| \leq 1 \tag{10.2.1}$$

By the same method as above we get the explicit expression of the support function $H_{D(T)}(\xi)$ of the attainable set $D(T)$ (from the origin) at time T

$$\begin{aligned} H_{D(T)}(\xi) &= \int_0^T |\xi_1 \cos t + \xi_2 \sin t| \, dt \\ &= T(\xi_1^2 + \xi_2^2)^{1/2} \int_0^1 \left|\cos T\left(t - \tfrac{\phi}{T}\right)\right| \, dt, \end{aligned} \tag{10.2.2}$$

where $\phi = \arctan\left(\frac{\xi_1}{\xi_2}\right)$.

Exercise 10.2.1 *Prove that* $\frac{H_{D(T)}(\xi)}{T}$ *tends to* $\frac{2}{\pi}|\xi| = \frac{2}{\pi}(\xi_1^2+\xi_2^2)^{1/2}$ *as* $T \to \infty$ *uniformly w.r.t.* $\xi, |\xi| = 1$. *Give a geometric interpretation of the result.*

Exercise 10.2.2 *Prove that if* T *is small, then the gradient map* $x(\xi) = \frac{\partial H_{D(T)}}{\partial \xi}$ *has multiple points, and that* $\partial D(T)$ *has corner points.*

Exercise 10.2.3 *Prove that if* T *is large, then the gradient map* $x(\xi) = \frac{\partial H_{D(T)}}{\partial \xi}$ *has no multiple points, and that* $\partial D(T)$ *is smooth. How large the* T *should be?*

General results on singularities of attainable sets and their asymptotic behavior see in references.

Chapter 11

White Noise I

11.1 Reminder on the Probability Theory

Mathematical probability theory takes the following standpoint on stochastic events: they are, in fact, deterministic, but depend on hidden (unknown) parameters. Thus, a random variable ξ is a function $\xi(\omega)$, $\omega \in \Omega$, where ω is a hidden parameter (elementary event) contained in the set Ω of all these parameters (space of elementary events). We are interested in some events which are collections of elementary events, and what we usually want to know is the probability of an event. The mathematical model of an event is a subset $A \subset \Omega$, and the probability is a function $P : A \mapsto P(A) \in [0, 1]$.

Precise definitions are as follows:

Definition 11.1.1 *The probability space is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a set named the space of elementary events (or simply probability space), $\mathcal{F}$ is a collection of subsets $A \subset \Omega$ (events), P is a function $P : \mathcal{F} \to [0, 1]$ (probability). These data are such that*

1. *$\mathcal{F}$ is a σ-algebra, meaning that if $A, B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$, $A \backslash B \in \mathcal{F}$, and $\bigcup_i A_i \in \mathcal{F}$ if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$*

2. *$\Omega \in \mathcal{F}$*

3. *$P(\bigcup_{i \in I} A_i) = \sum_i P(A_i)$ if $A_i \bigcap A_j =$ for $i \neq j$, I being any denumerable set of indices*

4. *$P(\Omega) = 1$*

If we omit P from the above definition, we get the definition of the *measurable space*. A map $f : \Omega \to \Omega'$ from one measurable space $(\Omega, \mathcal{F})$ to another $(\Omega', \mathcal{F}')$ is called measurable (denoted by $f \in \mathcal{F}$) iff $f^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{F}'$. A random element of a measurable space $(\Omega', \mathcal{F}')$ is by definition a measurable map f from a probability space $(\Omega, \mathcal{F}, P)$ to $(\Omega', \mathcal{F}')$. It induce a probability measure f_*P on $(\Omega', \mathcal{F}')$: $f_*P(A) = P(f^{-1}(A))$, which makes $(\Omega', \mathcal{F}', f_*P)$ a probability space. The measure f_*P is called the distribution of f, and is often denoted by P_f.
Any set Φ of subsets $A \subset \Omega$ belongs to the minimal σ-algebra containing it. This σ-algebra is denoted by $\sigma(\Phi)$ and called generated by Φ. Similarly, any set $F = \{f : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')\}$ of random elements determines the σ-algebra $\sigma(F) = \sigma(\{f^{-1}(A), A \in \mathcal{F}'\})$, which is also called the σ-algebra generated by F.

Example 11.1.2 *The most (at least historically) important probability space is* $([0,1], \mathcal{B}, dx)$, *where* $\Omega = [0,1]$, $\mathcal{B}$ *consists of the Borel sets (*σ*-algebra generated by open sets), and* dx *is the Lebesgue measure (length).*

Example 11.1.3 *Next example is the unit square* $([0,1]^2, \mathcal{B}, dxdy)$, *where* $\mathcal{B}$ *is again the Borel* σ*-algebra,* $dxdy$ *is the Lebesgue measure (area).*

Example 11.1.4 *The standard model of coin tossing is* $(\Omega, \mathcal{F}, P)$, *where* $\Omega = \{\epsilon_i, \epsilon_i \in \{0,1\}, i = \pm 1, \pm 2, \ldots\}$, $\mathcal{F}$ *is generated by the sets of the form* $\{\epsilon_k = 1\}$, *and* P *is such that* $P(\{\epsilon_{k_i} = \alpha_i, i \in I\}) = 2^{-|I|}$. *Thus, an elementary event is an infinite sequence (from* $-\infty$ *to* $+\infty$*) of tossing, the result being either* 0 *(head) or* 1 *(tail) with equal probability.*

Two probability spaces $(\Omega', \mathcal{F}', P')$ and $(\Omega'', \mathcal{F}'', P'')$ are isomorphic if there are sets $\widetilde{\Omega}' \in \mathcal{F}'$, $\widetilde{\Omega}'' \in \mathcal{F}''$, and a bijective measurable map $f : \widetilde{\Omega}' \to \widetilde{\Omega}''$ such that $P(\widetilde{\Omega}') = 1$, $P(\widetilde{\Omega}'') = 1$, $f^{-1}(A) \in \mathcal{F}'$ if $A \in \mathcal{F}''$, $A \subset \widetilde{\Omega}''$, and $P'(f^{-1}(A)) = P''(A))$. Two events $A, B \in \mathcal{F}$ are called equivalent if $P((A \backslash B) \bigcup (B \backslash A)) = 0$, and, similarly, two random elements are called equivalent if they differ on a set of the zero measure. Equivalent elements are indistinguishable from the experimental viewpoint and we will not usually distinguish between elements and their classes of equivalence.

Exercise 11.1.5 *Show that all the above examples are isomorphic.*

11.1.1 Measure and integral

A real random variable ξ is a random element $\xi : \Omega \to \mathbf{R}$ of $(\mathbf{R}, \mathcal{B})$, where $\mathbf{R}$ is the real line, and $\mathcal{B}$ is the σ-algebra of the Borel sets. The probability distribution P_ξ of ξ is simultaneously a (Schwartz) distribution on $\mathbf{R}$.

Mathematical expectation $\mathbf{E}\xi$ of ξ is by definition

$$\mathbf{E}\xi \stackrel{\text{def}}{=} \int_\Omega \xi(\omega) dP(\omega) = \int_{\mathbf{R}} x \, dP_\xi(x)$$

if ξ is an absolutely integrable function. Here, we will not describe the detail of construction of integral from measure. In rough outline we define $\int_\Omega 1_A(\omega) dP(\omega) = P(A)$, where 1_A is the indicator function of $A \in \mathcal{F}$, then extend it to finite linear combinations of the indicator functions, and pass to a limit.

11.1.2 Independence and conditional expectation

Two events $A, B \in \mathcal{F}$ are said to be independent if $P(AB) = P(A)P(B)$. Let $\mathcal{F}' \subset \mathcal{F}$ be a sub-σ-algebra. An event $A \in \mathcal{F}$ is said to be to be independent of $\mathcal{F}'$ if A, B are independent for any $B \in \mathcal{F}'$.

Example 11.1.6 *Consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0,1]^2$, $\mathcal{F}$ consists of the Borel sets (σ-algebra generated by open sets), and P is the Lebesgue measure (area). Put $\mathcal{F}'$ equal to the σ-algebra of sets of the form $A \times [0,1]$, where A is a Borel subset of $[0,1]$. Then a set $B \in \mathcal{F}$ of the form $[0,1] \times B'$, B' being a Borel subset of $[0,1]$, is independent of $\mathcal{F}'$.*

By a slight abuse of the language we say that a random element

$$f : (\Omega, \mathcal{F}) \to (\Omega'', \mathcal{F}'')$$

is independent of $\mathcal{F}'$ iff all the events from the σ-algebra $\sigma(f)$ generated by $f^{-1}(A), A \in \mathcal{F}''$ are independent of $\mathcal{F}'$. Two σ-algebras $\mathcal{F}', \mathcal{F}''$ are called independent provided that all events $A \in \mathcal{F}'$, $B \in \mathcal{F}''$ are independent. Two random variables ξ, η are called independent iff the σ-algebras $\sigma(\xi), \sigma(\eta)$ are independent. This condition is equivalent to $\mathbf{E}f(\xi)g(\eta) = \mathbf{E}f(\xi)\mathbf{E}g(\eta)$ for any couple of real measurable functions f, g.

Exercise 11.1.7 *Show that in the above example a random variable $f = f(x,y)$ is independent of $\mathcal{F}'$ iff the distribution of the random variable f_x :*

$y \mapsto f(x,y)$ does not depend on x almost surely (meaning that this is the case after removal of a set of measure zero from $[0,1]$).

Suppose we are given an integrable random variable ξ, $\mathbf{E}(|\xi|) < \infty$, and a σ-algebra $\mathcal{F}' \subset \mathcal{F}$. Then, there exists a unique (up to equivalence) $\mathcal{F}'$-measurable random variable $\mathbf{E}(\xi|\mathcal{F}')$ such that

$$\mathbf{E}(\mathbf{E}(\xi|\mathcal{F}')\eta) = \mathbf{E}(\xi\eta) \tag{11.1.1}$$

for any (bounded) $\mathcal{F}'$-measurable random variable η. The variable $\mathbf{E}(\xi|\mathcal{F}')$ is called the conditional expectation of ξ w.r.t. $\mathcal{F}'$.

Exercise 11.1.8 *Prove the uniqueness of $\mathbf{E}(\xi|\mathcal{F}')$, and show that its existence is implied by the following version of the Radom – Nicodim theorem:* Suppose, that P' is a probability measure on a probability space $(\Omega, \mathcal{F}, P)$, and it is such that $P'(A) = 0$ if $P(A) = 0$ for any $A \in \mathcal{F}$. Then, P' has the form

$$P'(A) = \int_A f(\omega) dP(\omega),$$

where f is a positive integrable random variable.

In the Radom – Nicodim setup the measure P' is called absolutely continuous w.r.t. P, and the function f is called the density of P' w.r.t. P.

Exercise 11.1.9 *Suppose that the above ξ is square integrable: $\mathbf{E}(|\xi|^2) < \infty$. Show that $\xi' = \mathbf{E}(\xi|\mathcal{F}')$ is the best $\mathcal{F}'$-measurable approximation to ξ in the L_2-sense: $\mathbf{E}(|\xi - \xi'|^2)$ is minimal possible.*

Interpretation of the statement 11.1.9 is as follows: The σ-algebra $\mathcal{F}'$ contains all available information. Say, $\mathcal{F}' = \sigma(\phi)$, where ϕ is a random variable that we can measure. Then $\mathbf{E}(\xi|\mathcal{F}')$ is the best possible prediction of the value of ξ that we can make on the basis of available data ϕ.

If $\Omega = \Omega' \times \Omega''$, $\mathcal{F} = \sigma(\mathcal{F}', \mathcal{F}'')$, and $dP(\omega', \omega'') = f(\omega', \omega'') dP'(\omega') dP''(\omega'')$, then $\mathbf{E}(\xi|\mathcal{F}'')$, where $\xi = \xi(\omega', \omega'')$, is given by

$$\mathbf{E}(\xi|\mathcal{F}'')(\omega'') = \frac{\int_{\Omega'} \xi(\omega', \omega'') f(\omega', \omega'') dP'(\omega')}{\int_{\Omega'} f(\omega', \omega'') dP'(\omega')}. \tag{11.1.2}$$

(Prove it!)

Exercise 11.1.10 *If a random variable ξ is independent of a σ-subalgebra $\mathcal{F}' \subset \mathcal{F}$, then $\mathbf{E}(\xi|\mathcal{F}')(\omega) = \mathbf{E}(\xi)$ for (almost) all ω.*

Exercise 11.1.11 *Extend the above results on conditional expectations of a real random variable to vector-valued variables.*

11.1.3 Characteristic functions and Gaussian vectors

The characteristic function of a real random variable ξ is a complex-valued function χ_ξ of the (deterministic!) real variable:

$$\chi_\xi(t) \stackrel{\text{def}}{=} \mathbf{E}\exp(it\xi) = \int_\Omega \exp(it\xi(\omega))dP(\omega) = \int_{\mathbf{R}} \exp(itx)\, dP_\xi(x)$$

which is the Fourier transform of the (Schwartz) distribution P_ξ. It determines the distribution P_ξ uniquely. The characteristic function of the sum of two independent random variables is the product of corresponding characteristic functions (prove it!).

The definition of the characteristic function has a natural extension to the case of random vectors. If $\xi \in V$ is a random vector, then its characteristic function χ_ξ is a function on the dual space V^* defined by

$$\chi_\xi(\lambda) = \mathbf{E}\exp(i\langle\lambda,\xi\rangle) = \int_V \exp(i\langle\lambda,x\rangle)\, dP_\xi(x)$$

One can characterize independence by using the characteristic functions:

Exercise 11.1.12 *Suppose that $\xi \in V$, $\eta \in W$ are random vectors and $\zeta = (\xi,\eta) \in V \oplus W$ is the compound vector. Then the vectors ξ, η are independent iff the characteristic function factorizes: $\chi_\zeta(z) = \chi_\xi(x)\chi_\eta(y)$ for $z = (x,y) \in V^* \oplus W^*$*

A distribution $p(x)$, $x \in \mathbf{R}$ is called Gaussian if it has the form $p(x) = \exp(Ax^2 + Bx + C)$, where $A, B, C \in \mathbf{C}$, or is a limit of such functions in the space of distributions. For instance, the Dirac δ-function is Gaussian (prove it!). One can characterize the Gaussian functions as solutions of nontrivial equations of the form

$$(\alpha\frac{\partial}{\partial x} + \beta x + \gamma)p(x) = 0,$$

where α, β, γ are (complex) constants (do this!).

A random variable is called Gaussian if its distribution is Gaussian.

Exercise 11.1.13 *Show that a random variable is Gaussian iff its characteristic function is Gaussian.*

The ubiquity of Gaussian variables is (partially) explained by the Central Limit Theorem:

Theorem 11.1.14 *Suppose that ξ_n is a sequence of independent identically distributed (real) random variables such that $\mathbf{E}\xi_n^2$ is finite. Put $m = \mathbf{E}\xi_n$, $\sigma = \mathbf{E}(\xi_n - m)^2$. Then, the distribution of the variable*

$$S_N \stackrel{\text{def}}{=} \frac{\sum_1^N (\xi_n - m)}{\sqrt{N}}$$

is approximately Gaussian: $P(S_N \in A) \to \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{1}{2\sigma}x^2} dx$ as $N \to \infty$ for any interval $A \subset \mathbf{R}$.

Proof (sketch). It suffices to show that the characteristic function $\chi_{S_N}(\lambda) \to e^{-\frac{1}{2}\sigma\lambda^2}$ as $N \to \infty$. But $\chi_{S_N}(\lambda) = \chi_{\xi'}(\frac{\lambda}{\sqrt{N}})^N$, where $\xi' = \xi_n - m$ (the distribution of ξ' does not depend on n). By the Taylor formula,

$$\chi_{\xi'}(\lambda) = 1 - \frac{1}{2}\sigma\lambda^2 + o(\lambda^2),$$

and therefore, $\lim_{N\to\infty} N \log \chi_{\xi'}(\frac{\lambda}{\sqrt{N}}) = -\frac{1}{2}\sigma\lambda^2$ which is what we need. ▲

Similarly to the scalar case we say that a distribution $p(x)$, $x \in \mathbf{R}^n$ is Gaussian if it has the form $p(x) = \exp(\langle Ax, x\rangle + \langle B, x\rangle + C)$, where A is a complex matrix, B is a complex vector, or is a limit of such functions in the space of distributions.

A vector-valued (with values in some $V = \mathbf{R}^n$) random variable ξ is called Gaussian if the real variable $\langle \lambda, \xi\rangle$ is Gaussian, where $\lambda \in V^*$ is any linear functional on V. An equivalent definition is that distribution of ξ is Gaussian, or that the characteristic function of ξ is Gaussian.

Exercise 11.1.15 (*not obvious*) *Find a characterization of the vector Gaussian distributions via differential equations.*

Solution: Suppose $V = \mathbf{R}^n$. Consider differential operators $M(\xi, \eta, \lambda)$, where $\xi, \eta \in \mathbf{C}^n$, and $\lambda \in \mathbf{C}$ of the form

$$M(\xi, \eta, \lambda) = \left\langle \xi, \frac{\partial}{\partial x} \right\rangle + \langle \eta, x\rangle + \lambda. \tag{11.1.3}$$

Take an n-dimensional subspace $\mathcal{L}$ in the $2n+1$-dimensional space $\mathfrak{h}$ of operators $M(\xi, \eta, \lambda)$ such that a) any pair of operators from $\mathcal{L}$ commute, b) nonzero constants do not belong to $\mathcal{L}$. Then, any distributional solution u of the system $Mu = 0$, where M runs over $\mathcal{L}$ is a Gaussian function. The converse is also true.

An intimately related fact is this:

Exercise 11.1.16 *Suppose, that f is a distribution in $V = \mathbf{R}^n$, and $\mathcal{L} \subset \mathfrak{h}$ consists of operators $M \in \mathfrak{h}$ such that $Mf = 0$. Then*

- $\dim_{\mathbf{C}} \mathcal{L} \leq n$
- *Any pair of operators from $\mathcal{L}$ commute*
- *Nonzero constants do not belong to $\mathcal{L}$*

Exercise 11.1.17 *Suppose that $\xi \in V$ is a V-valued Gaussian vector. Then, there exists a vector subspace $W \subset V$, a strictly positive definite quadratic form Q on the dual space W^*, and a vector $a \in V$ such that $\eta = \xi - a \in W$, and the distribution of η as a W-valued Gaussian vector has the (nondegenerate Gaussian) density $dP_\eta(x) = \det(2\pi Q)^{-1/2} e^{-\frac{1}{2}(Q^{-1}x,x)} dx$.*

Explanation of notations: If a nondegenerate quadratic form Q is given on W^* it naturally defines a quadratic form on its dual W. Indeed, a nondegenerate quadratic form identifies W^* and W. If W^* is identified with $\mathbf{R}^n$, then the quadratic form Q is identified with a symmetric matrix to be denoted by the same letter. Then the matrix of the quadratic form on the dual W is given by the matrix Q^{-1}.

Thus, a Gaussian vector ξ canonically determines a vector a_ξ in the space of possible values, and a Hilbert space structure on a subspace $W_\xi \in V$.

Vectors $\xi_i \in V_i$ are called jointly Gaussian if the compound vector

$$(\xi_1, \xi_2, \ldots) \in V_1 \times V_2 \times \ldots$$

is Gaussian. If ξ_i are jointly Gaussian, then any finite linear combination $\sum_i a_i \xi_i$, where $a_i \in \mathbf{R}$ is Gaussian (prove it!). The converse is also true (prove it!). If vectors ξ, ξ_i, $i = 1, 2, \ldots$ are jointly Gaussian, then the conditional expectation $X = \mathbf{E}(\xi | \sigma(\xi_1, \xi_2, \ldots))$ is Gaussian (prove it!). Basic fact is this:

Theorem 11.1.18 *Suppose, that ξ, η are centered (i.e., $\mathbf{E}\xi = 0$, $\mathbf{E}\eta = 0$) jointly Gaussian vectors. Then they are independent iff they are not correlated, meaning that $\mathbf{E}(\langle \xi, \lambda \rangle \langle \eta, \mu \rangle) = 0$, for any pair of linear functionals λ, μ.*

Proof. It suffices to show that $\langle \xi, \lambda \rangle$ and $\langle \eta, \mu \rangle$ are independent which reduces our task to the case of scalar centered Gaussian ξ, η such that $\mathbf{E}\xi\eta = 0$. The characteristic function of the Gaussian vector $\zeta = (\xi, \eta) \in \mathbf{R}^2$ has the form

$\chi_\zeta(z) = \exp(-\langle Qz, z\rangle + i\langle q, z\rangle)$ where Q is a symmetric 2×2 matrix, and q is a 2-vector. Since ξ, η are centered, we infer that $q = 0$, and from $\mathbf{E}\xi\eta = 0$ we infer that $Q_{12} = Q_{21} = \mathbf{E}\xi\eta = 0$. Therefore, $\chi_\zeta(z) = \chi_\xi(x)\chi_\eta(y)$ for $z = (x, y)$ which is equivalent to independence of ξ and η.

Exercise 11.1.19 *Put $\widehat{X} = X - \mathbf{E}X$ and define similarly $\widehat{\xi}$ and $\widehat{\xi}_i$. Prove that any component of the vector $\widehat{X}$ is a linear combination of components of the vectors $\widehat{\xi}_i$.*

Proof. Assume for simplicity that $\mathbf{E}\xi = 0 = \mathbf{E}\xi_i$ and drop the hat in what follows. We can also assume without loss of generality that ξ and ξ_i are scalar Gaussian variables. Then we will seek X in the form $X = \sum a_i\xi_i$, where $a_i \in \mathbf{R}$ and such that $\mathbf{E}\xi\xi_j = \sum_{i=1}^N a_i\mathbf{E}\xi_i\xi_j$ for any j. Indeed, then, in view the Theorem 11.1.18, the Gaussian variable $\xi - \sum a_i\xi_i$ is independent of all ξ_j. Geometrically, we just have to take the orthogonal projection of the vector $\xi \in L_2(\Omega, P)$ to the subspace generated by $\xi_i \in L_2(\Omega, P)$ thus obtaining X. This in fact completes the proof, but we present arguments of a non-geometric nature. In the matrix language, we are to solve the equation $Qa = b$, where $Q_{ij} = \mathbf{E}\xi_i\xi_j$ and $b_j = \mathbf{E}\xi\xi_j$. The criterion of solvability is this: if $Qc = 0$, then $\langle c, b\rangle = 0$. But one can see easily that $Qc = 0$ means that $\mathbf{E}\xi_i(\sum c_j\xi_j) = 0$ for any i. In particular, this implies $\mathbf{E}(\sum c_j\xi_j)^2 = 0$, and, therefore, $\sum c_j\xi_j = 0$. From this we immediately get that $\sum c_jb_j = \mathbf{E}(\sum c_j\xi_j)\xi = 0$.

Therefore, in general, forming the conditional expectation of ξ is an affine operation.

Chapter 12

White Noise II

12.1 Stochastic processes

There is no a consensual view of what a stochastic process is. There is, however, a common element in all definitions: the set of time instants T and the space of values $(\mathfrak{M}, \mathfrak{F})$.

We list some reasonable definitions:

1. A stochastic process is a set of random elements $\xi_t \in \mathfrak{M}$, where $t \in T$.
2. A stochastic process is a random element (sample path) in the space $\mathfrak{M}^T$ of maps $T \to \mathfrak{M}$.

The first definition is quite clear, the second is not, but the relation of the definitions is clear: if ξ is a random element of $\mathfrak{M}^T$, then $\xi_t \stackrel{\text{def}}{=} \xi(t)$ is well-defined. The second definition is not clear for two reasons: first, we have to specify a measurable structure on $\mathfrak{M}^T$ such that the map $\xi \to \xi(t)$ from $\mathfrak{M}^T$ to $\mathfrak{M}$ is measurable for any given $t \in T$, second, it is not always reasonable to consider all (sample) pathes $T \to \mathfrak{M}$. It might be better to deal with, say, measurable maps, or continuous maps, or analytic maps, or, on the contrary, some rather singular generalized functions. For example, if we want by some reason compute mathematical expectation of e^{-S}, where $S = \int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2)dt$ is the action for a harmonic oscillator. This makes no apparent sense if the path $t \to x(t)$ is not differentiable.

Another reasonable definition is this:

3. A stochastic process is a a probability measure P in the space $\mathfrak{M}^T$ of maps $T \to \mathfrak{M}$.

Again, this requires clarification, but the relation with the previous definition is as follows: $P(A) = \mathrm{Prob}(\xi \in A)$, where Prob is the probability measure involved into definition of the random element $\xi \in \mathfrak{M}^T$, A is a measurable subset of $\mathfrak{M}^T$. In other words, $P = \mathrm{Prob}_\xi = \xi_* \mathrm{Prob}$.

We have however an indisputable example of a random process — the coin tossing:

Example 12.1.1 $T = \{1, 2, \ldots\}$, $\mathfrak{M} = \{0, 1\}$, $\Omega = \mathfrak{M}^T$, $\xi_t(\omega) = \omega(t)$. *Here, T is the set of discrete time instants, point of $\mathfrak{M}$ is the result of a tossing at specific time instant, Ω is the set of all sequences $T \to \mathfrak{M}$ of all possible results of tossing. Independent random variables $\xi_t \in \{0, 1\}$ $t \in T$ are the results of tossing at time t. The σ-algebra $\mathcal{F}$, and measure P are defined in lecture 10. These data defines a stochastic process in all the above senses.*

In fact the above vague notion of stochastic process is not always adequate: one needs to consider generalized processes. Basically, these are the processes such that their sample pathes are generalized functions (distributions). We will consider only one-dimensional time: the set T will be an interval in the real axis. The set of values $\mathfrak{M}$ of a generalized process will be a finite dimensional (real) vector space: a proper definition of a generalized map to a nonlinear manifold is a great challenge. We denote by $\mathfrak{M}^*$ the dual space of $\mathfrak{M}$.

Again we list some reasonable definitions:

1. A generalized stochastic process ξ is a continuous linear map $\phi \to \langle \phi, \xi \rangle$ from the set of (deterministic) $\mathfrak{M}^*$-valued test functions to real random variables. Symbolically, $\langle \phi, \xi \rangle = \int \langle \phi(t), \xi(t) \rangle dt$.

2. A generalized stochastic process is a random element (sample path) in the space of generalized maps $T \to \mathfrak{M}$.

3. A generalized stochastic process is a a probability measure P in the space of generalized maps $T \to \mathfrak{M}$.

Now, all the above definitions require clarification in order to become mathematical ones. For instance, in the first definition one needs to choose a space of test functions, and define exactly what the adverb *continuous* means. In the second definition the space of generalized maps $T \to \mathfrak{M}$ is to be defined along with a measurable structure, and the same applies to the third definition. Still, there are some clear relations between the above definitions.

Say, a a random element ξ in the space of generalized maps $T \to \mathfrak{M}$ can be integrated against a test function ϕ in order to get a random variable $\langle \phi, \xi \rangle$, and an ordinary stochastic process defines a generalized one via $\langle \phi, \xi \rangle = \int \langle \phi(t), \xi(t) \rangle dt$.

In what follows we will deal a lot with Gaussian processes, which again have no a consensual definition, but instead a set of interrelated ones like the following:

1. A Gaussian process is a set of random jointly Gaussian vectors $\xi_t \in \mathfrak{M}$, where $t \in T$.

2. A Gaussian process is a Gaussian vector in the vector space $\mathfrak{M}^T$ of maps $T \to \mathfrak{M}$.

3. A Gaussian process is a Gaussian measure P in the vector space $\mathfrak{M}^T$ of maps $T \to \mathfrak{M}$.

4. A generalized Gaussian process ξ is a continuous linear map $\phi \to \langle \phi, \xi \rangle$ from the set of (deterministic) $\mathfrak{M}^*$-valued test functions to real Gaussian variables.

5. A generalized Gaussian process is a Gaussian vector in the vector space of generalized maps $T \to \mathfrak{M}$.

6. A generalized stochastic process is a Gaussian measure P in the vector space of generalized maps $T \to \mathfrak{M}$.

We will not clarify the above general definitions. Instead, we'll construct explicitly an interesting generalized Gaussian process — the white noise, which is one of the basic objects both in physics and mathematics.

12.2 Construction of the white noise

We will eventually build a generalized Gaussian process with independent values, defined at any time $t \in \mathbf{R}$, but we start with a periodic process in the interval $[0, 2\pi]$. We commence with a sequence of independent centered ($\mathbf{E} g_k = 0$) Gaussian complex variables g_k, where $k \in \mathbf{N}$ is a positive integer,

$\mathbf{E}\left((\operatorname{Re} g_k)(\operatorname{Im} g_k)\right) = 0$, $\mathbf{E}(\operatorname{Re} g_k)^2 = \frac{1}{2} = \mathbf{E}(\operatorname{Im} g_k)^2$, and put $g_{-k} = \bar{g}_k$, if $k \in \mathbf{N}$. Then we define

$$\xi(t) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbf{Z}} g_k e^{ikt} \tag{12.2.1}$$

and will try to make sense of the definition.

Of course, the sum does not converge. However, we can take a smooth real periodic function $\phi(t)$ and try to make sense of the formal integral

$$\int_0^{2\pi} \phi(t)\xi(t)dt = \sqrt{2\pi} \sum_{k \in \mathbf{Z}} g_k \phi_{-k}, \tag{12.2.2}$$

where ϕ_k is the kth Fourier coefficient of ϕ. One can see easily that the sum converge in $L_2(\Omega)$. Indeed, if $A \subset \mathbf{Z}$ is any subset such that $A = -A$, then

$$\mathbf{E} \left| \sum_{k \in A} g_k \phi_{-k} \right|^2 = \sum_{k \in A} \mathbf{E} |g_k \phi_{-k}|^2 = \sum_{k \in A} |\phi_{-k}|^2 \tag{12.2.3}$$

because of independence of g_k for $k \in \mathbf{N}$. In particular, if $\phi \in L_2([0, 2\pi])$, then the sum (12.2.2) converge in $L_2(\Omega, P)$, and it also follows from (12.2.3) that

$$\mathbf{E} \left(\int_0^{2\pi} \phi(t)\xi(t)dt \right)^2 = \int_0^{2\pi} \phi(t)^2 dt \tag{12.2.4}$$

Note, that the Gaussian properties of coefficients g_k played no role up to now. However, since the partial sums in (12.2.2) are obviously Gaussian, the formal integral (12.2.2) is Gaussian. Thus, we constructed a generalized Gaussian process in the interval $[0, 2\pi]$ according to the Definition 4 of a Gaussian process.

However, we do not know by now whether the formal sum (12.2.1) is a distribution, because this requires estimating the formal integral (12.2.2) pointwise as a function of $\omega \in \Omega$, not in $L_2(\Omega)$ as we did. For example, although the integral (12.2.2) is a well-defined random variable for any $\phi \in L_2([0, 2\pi])$, it is not true that $\xi(t)$ belongs to $L_2([0, 2\pi])$ with probability 1. In fact, this probability is 0 (prove it! Hint: this is equivalent to the fact, that

$$\sum_k |g_k|^2 = \infty$$

with probability 1. Use the law of large numbers.). We will show now that in fact, $\xi(t)$ is the derivative of a continuous function $w(t)$. In other words, we will show that

$$\sum_{0\neq k\in\mathbf{Z}} \frac{g_k}{ik} e^{ikt} \tag{12.2.5}$$

is the Fourier expansion of a continuous function with probability 1.

Define first

$$x_{mn}(t) = \sum_{k=m}^{n-1} \frac{g_k}{ik} e^{ikt}$$

and $s_{mn} = \sup_{t\in[0,2\pi]} |x_{mn}(t)|$. In what follows we will take indices m, n of the form $m = 2^k, n = 2^{k+1}$, $k \in \mathbf{N}$, and put $S_k = s_{mn}$, $X_k = x_{mn}$. Our basic estimate to be proved is

$$\mathbf{E}|S_k|^2 \leq C2^{-k/2}, \tag{12.2.6}$$

where C is an absolute constant. It implies immediately that the series

$$\sum_{l\in\mathbf{N}} X_l(t) \text{ which is formally equal to } \sum_{k\in\mathbf{N}} \frac{g_k}{ik} e^{ikt}$$

converges absolutely and uniformly in $[0, 2\pi]$ with probability 1 (provide detail!). Thus, the series defines a continuous function, and (12.2.5) is the Fourier expansion of a continuous function with probability 1.

To prove (12.2.6) we write down

$$|x_{mn}(t)|^2 = \sum_{k=m}^{n-1} \frac{|g_k|^2}{k^2} + 2\,\mathrm{Re} \sum_{l=1}^{n-m-1} e^{ilt} \sum_{j=m}^{n-l-1} \frac{g_j \bar{g}_{j+l}}{j(j+l)}, \tag{12.2.7}$$

which implies

$$|s_{mn}|^2 \leq \sum_{k=m}^{n-1} \frac{|g_k|^2}{k^2} + 2 \sum_{l=1}^{n-m-1} \left| \sum_{j=m}^{n-l-1} \frac{g_j \bar{g}_{j+l}}{j(j+l)} \right|. \tag{12.2.8}$$

We estimate the mathematical expectation of the inner sum $f_{ab} = \sum_{j=a}^{b-1} \frac{g_j \bar{g}_{j+l}}{j(j+l)}$ as follows:

$$(\mathbf{E}|f_{ab}|)^2 \leq \mathbf{E}|f_{ab}|^2 = \sum_{j=a}^{b-1} \frac{1}{j^2(j+l)^2} \leq \frac{b-a}{a^4} \tag{12.2.9}$$

because of independence of g_k (provide detail!). Therefore,

$$\mathbf{E}|s_{mn}|^2 \le \sum_{k=m}^{n-1} \frac{1}{k^2} + 2 \sum_{l=1}^{n-m-1} \left(\frac{n-l}{m^4}\right)^{1/2}, \tag{12.2.10}$$

and therefore,

$$\mathbf{E}|s_{m,2m}|^2 \le 3m^{-1/2}, \tag{12.2.11}$$

which implies (12.2.6) when $m = 2^k$. This completes the construction of the white noise in the interval $[0, 2\pi]$. What we have built is a Gaussian process according to all the above definitions (give detail!) and got a lot of extra information.

However, we need the white noise not in $T = [0, 2\pi]$ but in $T = \mathbf{R}$. This is easy to reach: For each $n \in \mathbf{Z}$ take an independent copy $\xi_n(t)$, $t \in [0, 2\pi]$ of the white noise just constructed, and then put $\xi(t) = \xi_n(t - \left[\frac{t}{2\pi}\right] 2\pi)$ (explain the last formula at the level of distributions!).

Now we list some basic properties of the white noise ξ:

1. ξ is a generalized real-valued centered Gaussian process in $T = \mathbf{R}$ according to all the above definitions.

2. $\xi(t)$ is the derivative of a continuous (ordinary) Gaussian process $w(t)$, $t \ge 0, w(0) = 0$. This latter process is called the Wiener process, or the Brownian motion.

3. $\xi(t)$ the process with independent values, meaning that the random variables $\langle \phi, \xi \rangle$ and $\langle \psi, \xi \rangle$ are independent, provided that the supports of the test functions ϕ, ψ are disjoint (i.e., $\phi\psi \equiv 0$).

4. We have for $\phi \in L_2(\mathbf{R})$

$$\mathbf{E}\left(\int_{\mathbf{R}} \phi(t)\xi(t)dt\right)^2 = \int_{\mathbf{R}} \phi(t)^2 dt \tag{12.2.12}$$

   which can be interpreted as

$$\mathbf{E}\xi(t)\xi(s) = \delta(t-s) \tag{12.2.13}$$

5. For any $\phi \in L_2(\mathbf{R})$

$$\mathbf{E}e^{i\int_{\mathbf{R}} \phi(t)\xi(t)dt} = e^{-\frac{1}{2}\int_{\mathbf{R}} \phi(t)^2 dt} \tag{12.2.14}$$

6. $\xi(t)$ is a stationary process, meaning that there is a one-parameter group T_t of measure preserving transformations of $(\Omega, \mathcal{F}, P)$ such that

$$\xi(t+s)(\omega) = \xi(t)(T_s(\omega)) \text{ for any } t, s.$$

(Deciphering: $T_t T_s = T_{t+s}$, $T_t(A) \in \mathcal{F}$ and $P(T_t(A)) = P(A)$ for any $A \in \mathcal{F}$)

Exercise 12.2.1 *Prove the above statements.*

Chapter 13

White Noise III

13.1 Why the white noise is white?

The white noise ξ can be characterized as a generalized Gaussian process such that for $\phi \in L_2(\mathbf{R})$

$$\mathbf{E}e^{i\int_{\mathbf{R}}\phi(t)\xi(t)dt} = e^{-\frac{1}{2}\int_{\mathbf{R}}\phi(t)^2dt}. \tag{13.1.1}$$

One of the interpretations of the property (13.1.1) is that the distribution P_ξ of ξ is the canonical centered Gaussian measure in the Hilbert space $L_2(\mathbf{R})$. This should not be taken literally, for the P_ξ-measure of $L_2(\mathbf{R})$ is zero, and sample pathes of ξ are generalized functions. However, it is true that any orthogonal transformation of $L_2(\mathbf{R})$ which can be continuously extended to the space of the Schwartz distributions (of polynomial growth) defines a measure preserving transformation. In particular, the Fourier transform

$$f \mapsto \widehat{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ist}f(t)dt$$

of $L_2(\mathbf{R})$ defines a new version $\widehat{\xi}$ of the initial white noise. The distribution $P_{\widehat{\xi}}$ of $\widehat{\xi}$ coincides with that of ξ.

To answer the title question we have to define the spectrum of a more or less arbitrary function, and declare a function representing white noise, if the spectrum is uniform in a suitable sense. We start with a finite superposition of harmonics

$$f(t) = \sum_\lambda a_\lambda e^{i\lambda t} \tag{13.1.2}$$

In this case we define the spectrum of f as the sum of δ-measures:

$$\mathrm{Spec}_f(u) = \sum_\lambda |a_\lambda|^2 \delta(\lambda - u). \tag{13.1.3}$$

In other words, Spec_f is a measure on $\mathbf{R}$ such that $\mathrm{Spec}_f(A) = \sum_{\lambda \in A} |a_\lambda|^2$.

One can see easily that the autocorrelation function $\phi_f(s) = \langle f(t+s)\overline{f(t)} \rangle$, where the angular brackets stand for averaging w.r.t., time has the form

$$\phi_f(s) \stackrel{\mathrm{def}}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+s)\overline{f(t)}dt = \sum_\lambda |a_\lambda|^2 e^{i\lambda s}. \tag{13.1.4}$$

Thus, it is the Fourier transform of Spec_f, and

$$\mathrm{Spec}_f(u) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iut} \phi_f(t) dt \tag{13.1.5}$$

One can also easily check that

$$\mathrm{Spec}_f(A) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_A \frac{1}{2\epsilon} \left| \int_{u-\epsilon}^{u+\epsilon} \widehat{f}(s) ds \right|^2 du \tag{13.1.6}$$

Exercise 13.1.1 *Prove the above formulae (13.1.4)–(13.1.6).*

Now we can state the whiteness of the white noise precisely in at least two ways:

$$\text{The autocorrelation function } \phi_\xi(s) = \delta(s) \text{ with probability 1.} \tag{13.1.7}$$

$$\text{The function } s_\epsilon(u) \stackrel{\mathrm{def}}{=} \tfrac{1}{2\epsilon} \left| \textstyle\int_{u-\epsilon}^{u+\epsilon} \widehat{\xi}(s) ds \right|^2 \text{ tends weakly to 1 with} \tag{13.1.8}$$

probability 1 as $\epsilon \to 0$.

Here, the autocorrelation function ϕ_ξ of the distribution ξ is defined as follows: First, we note that the distribution $\xi(t)\xi(s)$ of two variables is well-defined via

$$\int\int \xi(t)\xi(s)g(t)h(s)dtds = \int \xi(t)g(t)dt \int \xi(s)h(s)ds.$$

Then, we define $\phi_{\xi,T}$ via

$$\int \phi_{\xi,T}(u)g(u)du = \frac{1}{2T} \int\int \xi(t)\xi(s)1_{[-T,T]}(t)g(s-t)dtds,$$

and finally define $\phi_\xi = \lim_{T\to\infty} \phi_{\xi,T}$, if the limit exists in the topology of distributions.

We note first, that the statements (13.1.7), (13.1.8) are true at the level of mathematical expectations: Indeed,

$$\mathbf{E}\int \phi_{\xi,T}(u)g(u)du = \frac{1}{2T}\int\int \mathbf{E}(\xi(t)\xi(s))1_{[-T,T]}(t)g(s-t)dtds = \frac{1}{2T}\int\int \delta(t-s)1_{[-T,T]}(t)g(s-t)dtds = g(0)$$

which proves that $\mathbf{E}\phi_{\xi,T} = \delta$. Similarly,

$$\mathbf{E}s_\epsilon(u) = \mathbf{E}\frac{1}{2\epsilon}\left|\int_{u-\epsilon}^{u+\epsilon} \widehat{\xi}(s)ds\right|^2 = 1$$

for any $\epsilon > 0$.

Now we prove a weaker L_2-version of (13.1.8). Put $\sigma_\epsilon(u) = s_\epsilon(u) - 1$. We have to show that

$$\lim_{\epsilon\to 0}\mathbf{E}\left(\int_A \sigma_\epsilon(u)du\right)^2 = 0 \tag{13.1.9}$$

for any interval A. We divide A into sum of nonoverlapping intervals $A = \bigcup A_k$, $k = 1,\dots,N$ such that $\epsilon \le \text{length}(A_k) \le 2\epsilon$ and put $\eta_k = \int_{A_k}\sigma_\epsilon(u)du$. Then $\mathbf{E}\eta_k = 0$, $\mathbf{E}\eta_k^2 \ll |A_k|^2$, where $A \ll B$ stands for $A \le CB$ with an absolute constant C, and $\mathbf{E}\eta_k\eta_j = 0$ if $|k-j| > 1$ (prove it!). Therefore,

$$\mathbf{E}\left(\int_A \sigma_\epsilon(u)du\right)^2 = \mathbf{E}\left(\sum_{k=1}^N \eta_k\right)^2 \ll \sum_{k=1}^N |A_k|^2 \ll N\epsilon^2 \ll \epsilon$$

which proves (13.1.9).

We leave proving an L_2-version of (13.1.7) as an exercise for the listener.

13.2 Quantum mechanics and functional integrals

We turn to another naive interpretation of the basic property (13.1.1) of the white noise, which in a sense means that the distribution P_ξ is the canonical Gaussian distribution in the Hilbert space $L_2(\mathbf{R})$. It is also the canonical

Gaussian distribution in the Hilbert space $L_2([a,b])$ for any interval $[a,b]$. In a finite dimensional Hilbert space the canonical Gaussian distribution has the form

$$dP(x) = e^{-\frac{1}{2}|x|^2}dx$$

up to a constant multiplier, where $dx = dx_1 dx_2 \dots dx_N$ is the Lebesgue measure. This suggests that that we can think of P_ξ in a similar way, and that

$$dP_\xi(x) = e^{-\frac{1}{2}\int_a^b x(t)^2 dt} \prod_{t\in[a,b]} dx(t) \tag{13.2.1}$$

up to an irrelevant constant multiplier, where x is an arbitrary function from $L_2([a,b])$ and $\Pi_t dx(t)$ is a kind of the Lebesgue measure.

Another way to arrive at (13.2.1) is via approximation of the white noise by a piecewise constant random noise. Fix a large integer N and consider a piecewise constant centered Gaussian process $\xi_N(t) = \xi_N(\frac{1}{N}[Nt])$, $t \in [a,b)$ such that $\xi_N(\frac{k}{N})$ are independent for different integers k and $\mathbf{E}\xi_N(\frac{k}{N})^2 = N$. It is an approximation of the white noise from many points of view. In particular, the analogue of (13.2.1)

$$dP_{\xi_N}(x) = e^{-\frac{1}{2}\int_a^b x(t)^2 dt} \prod_{t=\frac{k}{N}\in[a,b)} \frac{dx(t)}{\sqrt{2\pi}}, \tag{13.2.2}$$

where x is an arbitrary piecewise (each piece being $[\frac{k}{N}, \frac{k+1}{N})$) constant function, holds true for the distribution of ξ_N in the space of piecewise constant functions $x(t)$ in the interval $[a,b)$ with ends of the form $\frac{k}{N}$ (prove it!).

The formula (13.2.1) takes even more suggestive form if we consider the Brownian motion $w(t)$ instead of the white noise $\xi(t) = \dot{w}(t)$. Namely,

$$dP_w(x) = e^{-\frac{1}{2}\int_a^b \dot{x}(t)^2 dt} \prod_{t\in[a,b]} \frac{dx(t)}{\sqrt{2\pi}} \tag{13.2.3}$$

and the factor $e^{-\frac{1}{2}\int_a^b \dot{x}(t)^2 dt} = e^{-S(x)}$, where $S(x)$ is the action of a free particle moving in one dimension along the trajectory $x(t)$. This resembles very much the Feynman approach to quantum mechanics via path integrals. His approach is not at all rigorous, but very suggestive. One of the basic formulas in the Feynman theory is this:

$$\exp(\frac{i}{\hbar}TH)(x,y) = \int_{\Omega(T,x,y)} e^{\frac{i}{\hbar}S(\gamma)} \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}} \tag{13.2.4}$$

Here, H is the Hamiltonian of a quantum system, $\exp(\frac{i}{\hbar}TH)$ is the operator of evolution, $\exp(\frac{i}{\hbar}TH)(x,y)$ is the integral kernel of this operator, and $\Omega(T,x,y)$ is the space of paths γ that start from x and end at y

$$\Omega(T,x,y) = \{\gamma : \gamma(0) = x,\ \gamma(T) = y\} \tag{13.2.5}$$

This can be restated in the form

$$\begin{aligned} &\exp(\tfrac{i}{\hbar}TH)f(x) \stackrel{\text{def}}{=} \textstyle\int \exp(\tfrac{i}{\hbar}TH)(x,y)f(y)dy = \\ &\textstyle\int_{\{\gamma(0)=x\}} e^{\frac{i}{\hbar}S(\gamma)} f(\gamma(T)) \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}}. \end{aligned} \tag{13.2.6}$$

The probabilistic counterpart of (13.2.4) is this:

$$\exp(-TH)(x,y) = \int_{\Omega(T,x,y)} e^{\hat{S}(\gamma)} \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}}, \tag{13.2.7}$$

where $\hat{S}(\gamma) = \int L(\gamma(t), i\dot{\gamma}(t))dt$, and $L(\gamma(t),\dot{\gamma}(t))$ is the Lagrangian of the quantum system. The counterpart of (13.2.6) is

$$\exp(-TH)f(x) = \int_{\{\gamma(0)=x\}} e^{\hat{S}(\gamma)} f(\gamma(T)) \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}}. \tag{13.2.8}$$

Consider now the case of a free particle, when

$$H = -\tfrac{1}{2}\left(\tfrac{\partial}{\partial x}\right)^2 \text{ and } \hat{S}(\gamma) = -\textstyle\int \frac{1}{2}\dot{x}(t)^2 dt.$$

In view of (13.2.1), the rigorous meaning of (13.2.8) should be

$$\exp(-TH)f(x) = \mathbf{E}f(x + w(T)). \tag{13.2.9}$$

This is equivalent to the fact that

$$\frac{\partial}{\partial T}\mathbf{E}f(x+w(T)) = \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 \mathbf{E}f(x+w(T)). \tag{13.2.10}$$

To verify (13.2.10) we put $y(s) = x + w(s)$, $\Delta = y(T+h) - y(T)$ and consider the Taylor expansion

$$f(y(T+h)) - f(y(T)) = \frac{\partial f}{\partial x}(y(T))\Delta + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(y(T))\Delta^2 + \dots \tag{13.2.11}$$

After observing that Δ is independent of $y(T)$, $\mathbf{E}\Delta = 0$, and $\mathbf{E}\Delta^2 = h$ we deduce that

$$\mathbf{E}(f(y(T+h)) - f(y(T))) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathbf{E}f(x + w(T))h + o(h)$$

which is equivalent to (13.2.10).

Exercise 13.2.1 *(The Feynman–Kac formula) Suppose that*

$$H = -\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 + V(x),$$

where V is a bounded continuous function. Then

$$\exp(-TH)f(x) = \mathbf{E}f(x(T))\exp\left(-\int_0^T V(x(s))ds\right),$$

where $x(t) = x + w(t)$. [Hint: Put $z(t) = \exp\left(-\int_0^t V(x(s))ds\right)$, and define $P_t f$ by $P_t f(x) = \mathbf{E}f(x(t))z(t)$. By using (13.2.11) and $z(T+h) - z(T) = V(x(T))z(T)h + 0(h)$ show that $\frac{\partial}{\partial t}P_t f(x) = P_t H f$ and deduce from this that $\frac{\partial}{\partial t}P_t f(x) = HP_t f$.]

Another, and more "reliable" way to prove (13.2.9) is as follows: We show first, that it suffices to check it for $f(x)$ of the form $e^{i\lambda x}$. Then, it remains to perform an easy calculation of both sides of (13.2.9) by using (13.1.1). We leave filling the detail as an exercise for the listener.

The right-hand side of (13.2.7) has the form

$$\int_{\Omega(T,x,y)} e^{\hat{S}(\gamma)} \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}} = e^{\hat{S}(\gamma_{x,y})} \int_{\Omega(T,0,0)} e^{\hat{S}(\gamma)} \prod_{s\in[0,T]} \frac{d\gamma(s)}{\sqrt{2\pi}}, \qquad (13.2.12)$$

where $\gamma_{x,y}$ is the straight line $\gamma_{x,y}(s) = x + \frac{s}{T}(y - x)$ connecting x with y, because $\hat{S}(\gamma_{x,y} + \gamma) = \hat{S}(\gamma) + \hat{S}(\gamma_{x,y})$, when $\gamma \in \Omega(T, 0, 0)$. Therefore, the right-hand side of (13.2.7) is $e^{-\frac{(x-y)^2}{2T}}C_T$, where C_T does not depend on x, y. Since $\int \exp(-TH)(x, y)dy = 1$ we obtain that $C_T = (2\pi T)^{-1/2}$, and, finally,

$$e^{-TH}(x, y) = \frac{e^{-\frac{(x-y)^2}{2T}}}{\sqrt{2\pi T}} \qquad (13.2.13)$$

which is a well-known (Poisson's) formula for the heat kernel.

Thus, the free quantum particle is in a sense a Brownian particle moving along the imaginary time axis.

13.3 Girsanov's theorem

Now we will use the basic formula (13.2.1) in order to get (rigorously) important properties of the white noise. First of them is the following formula for the change of the measure P_ξ under the shift $\xi(t) \mapsto \eta(t) = \xi(t) + h(t)$ in the Hilbert space $L_2([a,b])$:

$$dP_\eta(y) \stackrel{\text{def}}{=} dP_\xi(y-h) = \exp\left(-\frac{1}{2}\int_a^b h^2 dt + \int_a^b hy dt\right) dP_\xi(y) \qquad (13.3.1)$$

which is quite obvious after substitution $x(t) = y(t) - h(t)$ in (13.2.1). Rigorously, (13.3.1) means the following.

Theorem 13.3.1 *(Cameron – Martin – Girsanov theorem) Suppose that $h \in L_2([a,b])$ and f is a measurable bounded functional in the space of Schwartz distributions such that it depends only on restriction of a distribution on $[a,b]$. Then,*

$$\int f(x+h) dP_\xi(x) = \int \exp\left(-\frac{1}{2}\int_a^b h^2 dt + \int_a^b hy dt\right) f(y) dP_\xi(y). \quad (13.3.2)$$

Proof (sketch). It suffices to consider functionals of the form

$$f(y) = \exp(i\langle \phi, y\rangle),$$

where ϕ is a test function. For f of this form the result follows from (13.1.1) (provide detail!).

Chapter 14

Kalman Filter I

14.1 Equations for the conditional density

We study a two-component stochastic process governed by

$$\dot{x} = Ax + B\xi, \tag{14.1.1}$$

where ξ is a white noise. And we observe a process y related to x via

$$y = Cx + \eta,\ y(0) = 0, \tag{14.1.2}$$

where η is another white noise independent of ξ. We are interested in the evolution of the conditional density $p(t,x)$ for the process x. This density is defined via

$$\int u(x)p(t,x)\,dx = \mathbf{E}(u(x(t))|\mathcal{F}_t), \tag{14.1.3}$$

where the σ-algebra $\mathcal{F}_t$ is generated by $y(\tau)$, $\tau \in [0,t]$. In other words, if we denote by y^t the process y restricted to $[0,t]$, then

$$\mathbf{E}\left(\left(\int u(x)p(t,x)\,dx\right)\,v(y^t)\right) = \mathbf{E}\left(u(x(t))v(y^t)\right), \tag{14.1.4}$$

where $v(y^t)$ is any measurable functional.

Now we invoke the Girsanov theorem 13.3.1 about dependence of the probability measure on pathes $[0,T] \ni t \mapsto y(t)$ on the drift $h = Cx$. Namely, denote this measure by P^h. The measure $P = P^0$ is the Wiener (white noise) measure. The Girsanov formula tells:

$$\frac{dP^h(y)}{dP(y)} = \exp\left(-\frac{1}{2}\int_0^T |h|^2(s)ds + \int_0^T \langle h(s), y(s)\rangle\right), \tag{14.1.5}$$

where $h(s)$ is a shortening for $h(x(s))$.

Denote the right-hand side of (14.1.5) by $\phi(T) = \phi(T, x^T, y^T)$, where x^T, y^T are trajectories of our diffusion processes in the time interval $[0, T]$, and by $\mathcal{P}$ denote the measure of the process x. One sees immediately (see lecture 10 equation (1.2)) that the conditional density $p(t, x) = p(t, x, y^t)$ is defined by

$$\int u(x)p(t,x)dx = \frac{\int u(x(t))\phi(t, x^t, y^t)d\mathcal{P}(x^t)}{\int \phi(t, x^t, y^t)d\mathcal{P}(x^t)} \tag{14.1.6}$$

We note first that if we know the conditional density $p(t, x)$ up to a scalar factor $\lambda(t)$, then we can restore $p(t, x)$ due to normalization $\int p(t, x)dx = 1$. Second, we note that that one can replace $\mathcal{P}(dx^t)$ in both numerator and denominator integral in (14.1.6) with $\mathcal{P}(dx^T)$, where T is any time $\geq t$. Indeed, the drift coefficient corresponding to times $\geq t$ does not affect $x(t)$ and y^t, and, therefore, the conditional probability $p(t, x)$.

Therefore, it suffices to find the conditional density up to a constant, and, thus, it suffices to find $\rho(t, x)$ defined by

$$\int u(x)\rho(t,x)dx = \int u(x(t))\phi(t, x^t, y^t)d\mathcal{P}(x^T) \tag{14.1.7}$$

because $p(t, x) = \rho(t, x)/\lambda(t)$, where $\lambda(t) = \int \phi(t, x^t, y^t)d\mathcal{P}(x^T)$. In other words, $\int u(x)\rho(t, x)dx$ is equal to

$$\int u(x(t)) \exp\left(-\frac{1}{2}\int_0^t |h|^2(s)ds + \int_0^t \langle h(s), y(s)\rangle\right) d\mathcal{P}(x^t), \tag{14.1.8}$$

or what is equivalent

$$\mathbf{E}\left(u(x(t)) \exp\left(-\frac{1}{2}\int_0^t |Cx|^2(s)ds + \int_0^t \langle Cx(s), y(s)\rangle\right)\right). \tag{14.1.9}$$

In a sense the preceding formula gives an explicit expression for $\rho(t, x)$, which is, however, rather impractical. There are, however, at least two ways to deal with (14.1.8): 1) rewrite it as a functional integral obtained by quantization of a classical Hamiltonian system, and then try to compute it in classical terms, 2) find an evolutional equation for $\rho(t, x)$ and solve it. In fact, both approaches are intimately related. In this lecture we will follow the second path.

The Zakai equation

Let us study a simpler Kolmogorov's equation which is the evolutional equation for (conditional) density $p(t,x)$ when there are no condition (observation). The density $p(t,x)$ is defined via

$$\int_{\mathbf{R}^n} p(T,x)u(x)dx = \mathbf{E}u(x(T)). \tag{14.1.10}$$

We will need the infinitesimal operator $\mathcal{L}$ of the process x defined via

$$\mathcal{L} \stackrel{\text{def}}{=} \left\langle f, \frac{\partial}{\partial x} \right\rangle + \left\langle g^* \frac{\partial}{\partial x}, g^* \frac{\partial}{\partial x} \right\rangle, \tag{14.1.11}$$

where $f(x) = Ax$, $g(x) = B$, and the bracketed expressions are $\sum_i f_i \frac{\partial}{\partial x_i}$, resp. $\frac{1}{2}\sum_j \left(\sum_i g_{ij} \frac{\partial}{\partial x_i}\right)^2$.

Theorem 14.1.1

$$\frac{\partial}{\partial t} p(t,x) = \mathcal{L}^* p(t,x), \tag{14.1.12}$$

where $\mathcal{L}^*$ is the adjoint operator for $\mathcal{L}$.

Proof. (At Physical level of rigor) The statement is equivalent to

$$\frac{\partial}{\partial T} \mathbf{E}u(x(T)) = \mathbf{E}\mathcal{L}u(x(T)) \tag{14.1.13}$$

for any smooth function u with compact support. The subsequent arguments are based on the idea that the process $x(t)$ has increments of order $h^{1/2}$ of magnitude in time h, therefore to compute $u(x(t+h))$ with accuracy $o(h)$ one has to consider first *two* terms of the Taylor expansion of u at point $x(t)$.

Now we consider the finite difference

$$\mathbf{E}(u(x(T+h)) - u(x(T))) \tag{14.1.14}$$

and expand the integrand into the Taylor series:

$$u(x(T+h)) - u(x(T)) = \langle \frac{\partial u}{\partial x}, \Delta x \rangle + \frac{1}{2}\langle \frac{\partial^2 u}{\partial x^2}\Delta x, \Delta x\rangle + O(\Delta x^3), \tag{14.1.15}$$

where derivatives are taken at $x(T)$ and $\Delta x = x(T+h) - x(T)$. Now, in view of (15.1.1) we have $x(T+h) - x(T) = A(T)x(T) + \Delta + o(h)$, where the vector Δ is defined as

$$\Delta = \int_T^{T+h} \Phi(T+h,s)B(s)\xi(s)ds,$$

where $\Phi(T+h,s)$ is the fundamental matrix of the linear system $\dot{x}=Ax$. The most important properties of Δ are the following:

1. Δ is independent of the σ-algebra $\mathcal{F}_T$ generated by $\xi(s)$, $s\le T$. In particular, it is independent of $\frac{\partial u}{\partial x}(x(T))$ and $\frac{\partial^2 u}{\partial x^2}(x(T))$

2. $\mathbf{E}\Delta_i\Delta_j = h(BB^*)_{ij}(T)+o(h)$

(prove it!) By using the above properties of Δ, we infer from (14.1.15) that

$$\mathbf{E}(u(x(T+h))-u(x(T))) = h\mathbf{E}\left(\left\langle\frac{\partial u}{\partial x},Ax\right\rangle+\frac{1}{2}\left\langle\frac{\partial^2 u}{\partial x^2}BB^*\right\rangle\right)+O(\Delta^3)+o(h) \tag{14.1.16}$$

which implies (14.1.13) ▶

Now, we proceed to the evolutional equation for the conditional pseudo-density $\rho(t,x)$.

Theorem 14.1.2 (the Zakai theorem [22])

$$\frac{\partial}{\partial t}\rho(t,x) = \mathcal{L}^*\rho(t,x)-\frac{1}{2}|Cx|^2\rho(t,x)+(Cx,y)\rho(t,x), \tag{14.1.17}$$

where $\mathcal{L}^*$ is the adjoint operator for $\mathcal{L}$.

Proof is similar to the preceding one (however it is harder to make it rigorous) and the primary part of it is left to the reader. The statement is equivalent to

$$\begin{array}{l}\frac{\partial}{\partial t}\mathbf{E}\left(u(x(t))\exp\left(-\frac{1}{2}\int_0^t|Cx|^2(s)ds+\int_0^t\langle Cx(s),y(s)\rangle\right)\right)= \\ \mathbf{E}\left(\mathcal{L}-\frac{1}{2}|Cx(t)|^2+(Cx(t),y(t))\right)u(x(t))\end{array} \tag{14.1.18}$$

for any smooth function u with compact support. ▶
We will call the operator

$$Z=\mathcal{L}^*-\frac{1}{2}|Cx|^2+(Cx,y) \tag{14.1.19}$$

in the right-hand side of (14.1.17) the Zakai operator.

14.2 The Kalman filter

The problem resolved by Kalman and Bucy was about filtering (Gaussian) diffusion processes governed by linear equations:

$$\begin{aligned} \dot{x} &= Ax + B\xi \\ y &= Cx + \eta \end{aligned} \tag{14.2.1}$$

where ξ, η are independent white noises, and the matrices A, B, C might depend on time. The initial value $x(0)$ is a Gaussian random vector. The process y is observed, and one has to find the conditional distribution $p(x) = p(t,x)$ of $x(t)$ if the (generalized) sample path y^t of the process y up to time t is known. Since the conditional distribution is Gaussian, then

$$p(x) = C' \exp\left(-\frac{1}{2}\langle R^{-1}(x-m), x-m\rangle\right), \tag{14.2.2}$$

where $C' = (\det 2\pi R)^{-1/2}$ is a well-known irrelevant constant, $m = \mathbf{E}(x(t)|y^t)$ is the conditional expectation of $x(t)$, the conditional covariance matrix R is defined by $\langle R\xi, \eta\rangle = \mathbf{E}\left(\langle x(t)-m, \xi\rangle\langle x(t)-m, \eta\rangle\right)$. All quantities involved might depend on time t.

It is clear that the determination of (the evolution of) $p(x)$ is equivalent to the determination of m and R. We proceed from the Zakai equations (14.1.17). The Zakai operator Z from (14.1.19) takes the form:

$$Z = -\left\langle Ax, \frac{\partial}{\partial x}\right\rangle + \frac{1}{2}\left\langle BB^* \frac{\partial^2}{\partial x^2}\right\rangle - \frac{1}{2}\langle C^*Cx, x\rangle - \operatorname{Tr} A + \langle Cx, y\rangle \tag{14.2.3}$$

What is very special about the Zakai operator Z is that it naturally operates on the Heisenberg algebra.

14.2.1 Heisenberg algebra

This is the Lie algebra $\mathfrak{h}$ of differential operators $M(\xi, \eta, \lambda)$, where $\xi, \eta \in \mathbf{R}^n$, and $\lambda \in \mathbf{R}$ of the form

$$M(\xi, \eta, \lambda) = \left\langle \xi, \frac{\partial}{\partial x}\right\rangle + \langle \eta, x\rangle + \lambda \tag{14.2.4}$$

which act on, say, smooth functions of $x \in \mathbf{R}^n$. Here, Lie algebra means exactly that the commutator of two operators from $\mathfrak{h}$ is again in $\mathfrak{h}$. Indeed, the commutator is given by

$$[M(\xi', \eta', \lambda'), M(\xi'', \eta'', \lambda'')] = M(0, 0, \omega((\xi', \eta'), (\xi'', \eta''))). \tag{14.2.5}$$

Here, ω is a symplectic form on the space $\mathbf{R}^{2n}$ given by

$$\omega((\xi', \eta'), (\xi'', \eta'')) = \langle \xi', \eta'' \rangle - \langle \xi'', \eta' \rangle.$$

The differential operators of the form

$$\left\langle \xi', \frac{\partial}{\partial x} \right\rangle \left\langle \xi'', \frac{\partial}{\partial x} \right\rangle, \ \langle \eta', x \rangle \langle \eta'', x \rangle, \ \langle \eta', x \rangle \left\langle \xi'', \frac{\partial}{\partial x} \right\rangle \tag{14.2.6}$$

normalize the Heisenberg algebra, meaning that if operator A belongs to the list (14.2.6), and $B \in \mathfrak{h}$, then the commutator $[A, B] \in \mathfrak{h}$. This immediately implies that the Zakai operator Z from (14.2.3) normalizes $\mathfrak{h}$. Suppose, that an operator A normalizes $\mathfrak{h}$. Define $\tau(A) : \mathfrak{h} \to \mathfrak{h}$ by $\tau(A)B = [A, B]$ (adjoint representation). One can easily give an explicit form of the operator $\tau(Z)$, where Z is the Zakai operator (14.2.3).

Namely, introduce the matrix

$$\mathcal{Z}_0 = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix} \tag{14.2.7}$$

Then,

$$\tau(Z)M(\xi, \eta, \lambda) = M(\xi', \eta', -\langle \xi, C^*y \rangle) \tag{14.2.8}$$

where

$$(\xi', \eta')^* = \mathcal{Z}_0(\xi, \eta)^*. \tag{14.2.9}$$

We note, that the matrix $\mathcal{Z}_0$ is infinitesimally symplectic, meaning that

$$\omega(\mathcal{Z}_0 u, v) + \omega(u, \mathcal{Z}_0 v) = 0 \text{ for any } u, v \in \mathbf{R}^{2n}.$$

Moreover, this is the matrix of a linear Hamiltonian system with the Hamiltonian function

$$\mathfrak{H}(\xi, \eta) = \langle A\xi, \eta \rangle + \frac{1}{2}|B^*\eta|^2 - \frac{1}{2}|C\xi|^2.$$

Another important for us aspect of the Heisenberg algebra is that it is intimately connected with the Gaussian functions. Namely, a trivial computation shows that $M(\xi,\eta,\lambda)p$, where p is the Gaussian function (14.2.2), is given by

$$M(\xi,\eta,\lambda)p(x) = \left(\langle x, \eta - R^{-1}\xi\rangle + \langle m, R^{-1}\xi\rangle + \lambda\right) p(x). \tag{14.2.10}$$

In particular, $M(\xi,\eta,\lambda)p = 0$ is equivalent to two conditions

$$\begin{array}{rcl} \xi &=& R\eta, \\ \lambda &=& -\langle m, \eta\rangle \end{array} \tag{14.2.11}$$

Conversely, if a function f satisfy $M(\xi,\eta,\lambda)f = 0$ for each triple ξ,η,λ such that the condition (14.2.11) holds, then f coincides with the Gaussian function (14.2.2) up to a constant multiplier.

14.2.2 Lax type equations

Now our problem is this: We are looking for a Gaussian solution $\rho(t,x)$ of

$$\dot\rho = Z\rho \tag{14.2.12}$$

Note, that Gaussian here means *proportional*, not necessarily equal to a Gaussian probability distribution. We will indicate the dependence of different objects of time by the subscript t, and suppose that $M_0\rho_0 = 0$, where $M_t = M(\xi_t,\eta_t,\lambda_t)$. If M is a solution of

$$\dot M = [Z, M], \tag{14.2.13}$$

then one can easily check that $\phi = M\rho$ satisfies

$$\dot\phi = Z\phi. \tag{14.2.14}$$

Since $\phi_0 = 0$, we obtain that

$$M\rho = 0 \text{ at any time } t. \tag{14.2.15}$$

Now, we can combine these arguments with that of the preceding subsection, and conclude that if $M_t = M(\xi_t,\eta_t,\lambda_t)$ is the solution of the linear equation

$$\dot M = \tau(Z)M \tag{14.2.16}$$

on the Heisenberg algebra, and if

$$\xi_0 = R_0\eta_0, \langle m_0, \eta_0\rangle + \lambda_0 = 0, \tag{14.2.17}$$

then

$$\xi_t = R_t\eta_t, \langle m_t, \eta_t\rangle + \lambda_t = 0 \tag{14.2.18}$$

at any time t. It remains to see that equations (14.2.16), (14.2.18) together with (14.2.8), (14.2.9) determine the Kalman filter.

14.2.3 The Riccatti equation

We have to differentiate the first equation (14.2.18) w.r.t. time t. This gives

$$\dot{\xi} = \dot{R}\eta + R\dot{\eta}, \tag{14.2.19}$$

while

$$\dot{\xi} = A\xi + BB^*\eta = (AR + BB^*)\eta,$$

and

$$\dot{\eta} = C^*C\xi - A^*\eta = (C^*CR - A^*)\eta$$

by virtue of (14.2.8) and (14.2.18).

Combining the above equations, we get

$$(AR + BB^*)\eta = \dot{R}\eta + R(C^*CR - A^*)\eta$$

for any η. In other words,

$$\dot{R} = AR + RA^* + BB^* - RC^*CR, \tag{14.2.20}$$

which is the celebrated Riccatti equation. Its salient feature is that it does not depend on observations.

Similarly, by differentiation of the second equation (14.2.18), we get

$$\begin{aligned} 0 &= \langle \dot{m}, \eta\rangle + \langle m, \dot{\eta}\rangle + \dot{\lambda} \\ &= \langle \dot{m}, \eta\rangle + \langle m, (C^*CR - A^*)\eta dt\rangle - \langle \xi, C^*dy\rangle \\ &= \langle \dot{m}, \eta\rangle + \langle m, (C^*CR - A^*)\eta\rangle - \langle R\eta, C^*y\rangle, \end{aligned}$$

or what is the same

$$\dot{m} = (A - RC^*C)m + RC^*y. \tag{14.2.21}$$

Some people prefer another form of this equation:

$$dm = Am\,dt + RC^*(y - Cm)dt, \tag{14.2.22}$$

because $(y - Cm)dt$ is the differential of a new Wiener process (innovation process).

Equations (14.2.20) and (14.2.21) are the classical Kalman – Bucy equations from [16].

Chapter 15

The Kalman Filter II

15.1 The filtering problem

We study a two-component stochastic process governed by

$$\dot{x} = Ax + B\xi, \tag{15.1.1}$$

where ξ is a white noise. And we observe a process y related to x via

$$y = Cx + \eta,\ y(0) = 0, \tag{15.1.2}$$

where η is another white noise independent of ξ. We assume, that the matrices are locally-bounded measurable functions of time. Our main task here is to describe the evolution of the conditional mean of the unobservable vector $m_t = \widehat{x}_t = \mathbf{E}(x_t|y_0^t)$, where y_0^t stands for the observed trajectory $\tau \mapsto y(\tau)$, $\tau \in [0, t]$.

15.1.1 Innovation process

We invoke an important notion of the innovation process: the stochastic process

$$z_t = y_t - C_t m_t = C_t(x_t - m_t) + \eta_t. \tag{15.1.3}$$

This process possesses two basic properties:

1. This is a white noise.
2. The knowledge of trajectory z_0^t contains the same information as the knowledge of y_0^t.

To establish these properties we need some preparation. For starters, it is clear that if one knows y_0^t, then, the m_t is also known as well as $z_t = y_t - C_t m_t$. Thus, y_0^t contains no less information than z_0^t.

All the processes involved are Gaussian. Moreover, we can assume that these processes are also centered. This implies that the conditional mean m_t is given by a linear integral functional of y_0^t, i.e.,

$$m_t = \int_0^t K(t,s) y_s ds, \text{ where } \int_0^t |K(t,s)|^2 ds < \infty \tag{15.1.4}$$

for some deterministic square-integrable kernel $K(t,s)$. Here, $K(t,s)$ is a matrix, and $|K|^2 = \mathrm{Tr}(KK^*)$. Moreover, for any finite $T > 0$ the norm given by $\|K\|^2 = \|K\|_T^2 = \sup_{t\in[0,T]} \int_0^t |K(t,s)|^2 ds$ is finite.

Exercise 15.1.1 *Prove the above statement (15.1.4).*

Hint: Derive from the decomposition $y = Cx + \eta$, where x and η are independent, that $\mathbf{E}|\int \langle k(s), y(s)\rangle ds|^2 \geq \int_0^t |k(s)|^2 ds$ for any deterministic function k.

Therefore, the processes y_t, z_t are related by the Volterra equation

$$z_t = y_t - \int_0^t C(t) K(t,s) y_s ds \tag{15.1.5}$$

with a square-integrable kernel. One can, however, find a unique solution y of the above Volterra equation

$$y_t = z_t + \int_0^t \widetilde{K}(t,s) z_s ds, \tag{15.1.6}$$

where $\widetilde{K}(t,s)$ is another square-integrable kernel.

Exercise 15.1.2 *Prove the inversion formula (15.1.6).*

Hint: One can write (15.1.5) in the form $z = (1-K)y$, where K is the Volterra integral operator with kernel $k(t,s) = C(t)K(t,s)$. Formally,

$$y = (1-K)^{-1} z = (\sum K^n) z$$

and it remains to prove, that the Neumann series $\sum K^n$ converges in the space of Volterra operators with square-integrable kernels. But, the kernel

of K^n is $k_n(t,s) = \int k(t,s_1)k(s_1,s_2)\cdots k(s_{n-1},s)ds_1\cdots ds_{n-1}$, where the integral is over $\{(s_1,\ldots,s_{n-1}) : t \geq s_1 \geq \ldots s_{n-1} \geq s\}$ and the squared norm $\int_0^t |k_n(t,s)|^2 ds$ is less than

$$A_n^2 = \frac{(ct)^{n-1}}{(n-1)!}\int |k(t,s_1)|^2|k(s_1,s_2)|^2\cdots|k(s_{n-1},s)|^2 ds_1\cdots ds_{n-1}ds,$$

where $c = c_n$ is a constant. It is easy to see that if $t \leq T$, then

$$A_n^2 \leq \frac{(cT)^{n-1}}{(n-1)!}\|k\|_T^{2n},$$

so that the series $\sum A_n$ converges.

In particular, it follows from (15.1.5), (15.1.6) that trajectories z_0^t and y_0^t contain the same amount of information.

Now, consider the covariance $R_{zz}(t,s) = \mathbf{E}(z_t \otimes z_s)$. Here, we use the notation $x \otimes y$ for the matrix with components $(x \otimes y)_{ij} = x_i y_j$.

Exercise 15.1.3 *Prove that* $(Ax)\otimes y = A(x\otimes y)$, *and* $x\otimes(By) = (x\otimes y)B^*$.

We have for $t \geq s$ that

$$R_{zz}(t,s) = \mathbf{E}\left((C_t(x_t - m_t) + \eta_t)\otimes z_s\right) = \mathbf{E}\left(\eta_t \otimes z_s\right) = \mathbf{E}\left(\eta_t \otimes \eta_s\right) = \delta(t-s). \tag{15.1.7}$$

Indeed, $x_t - m_t$ is not correlated with y_s for $s \leq t$, which proves the second identity. The random variable η_t is independent of ξ_s and η_s for $s < t$. This implies, that η_t is independent of $x_s - m_s$ if $s < t$. By continuity the same holds for $s = t$, and this explains the third identity. The last identity holds, since η_t is a white noise.

Exercise 15.1.4 *Why the continuity arguments do not prove that* η_t *and* η_s *are always independent?*

It follows from (15.1.7), that the generalized stochastic process z_t is a white noise.

15.1.2 Equations for the conditional mean

Consider a random vector X which is jointly Gaussian with the process $y(t)$. One can write a canonical representation of the conditional expectation

$M = \mathbf{E}(X|y_0^t)$ as

$$M = \int_0^t K(s)z(s)ds, \text{ where } K(s) = \mathbf{E}(X \otimes z(s)). \tag{15.1.8}$$

Indeed, if M is given by the above formula, then $X - M$ is not correlated with z_s, provided that $0 \leq s \leq t$. Here, $K(s)$ is a square-integrable function:

$$\int_0^t K(s)K(s)^* ds = \mathbf{E}|M|^2 \leq \mathbf{E}|X|^2. \tag{15.1.9}$$

Exercise 15.1.5 *Prove the above formula.*

If we deal with a jointly Gaussian process $X(t), y(t)$ one can rewrite the identity (15.1.8) in the form

$$M_t = \mathbf{E}(X_t|y_0^t) = \int_0^t K(t,s)z(s)ds, \text{ where } K(t,s) = \mathbf{E}(X(t) \otimes z(s)). \tag{15.1.10}$$

The latter formula is meaningful even if the process X_t is generalized. Then, the kernel $K(t,s)$ is a distribution wrt the argument s.

Exercise 15.1.6 *Make the preceding statement about generalized processes precise and prove it.*

The conditional expectation m_t of the unobservable process x_t should satisfy an equation of the form

$$\dot{m} = Am + Dz, \tag{15.1.11}$$

where the matrix $D = D_t$ is to be found. Indeed, the process

$$n(t) = \dot{m}(t) - Am(t)$$

is the conditional mathematical expectation

$$n(t) = \mathbf{E}(\dot{x}(t) - Ax(t)|y_0^t) = \mathbf{E}(B(t)\xi(t)|y_0^t)$$

of the process $\nu(t) = B(t)\xi(t)$. The process ν_t is independent of y_0^s for $t > s$ due to the fact, that the white noise ξ_t is independent of ξ_0^s and η_0^s. Therefore, in the canonical representation $\mathbf{E}(\nu_t|y_0^t) = \int_0^t K(t,s)z(s)ds$ the Schwartz distribution $K(s) = s \mapsto K(t,s)$ is supported in t. This means in general, that $K(s)$ is a finite sum

$$K(s) = \sum a_k(\partial/\partial s)^k \delta(s-t), \tag{15.1.12}$$

where a_k are some matrices.

Exercise 15.1.7 *Prove the structural formula (15.1.12) for a Schwartz distribution such that its support is contained in* $\{t\}$.

In our particular situation, the distribution $K(s)$ is not very singular. More precisely, it is a derivative of a square-integrable function, because the generalized process ν_t is a derivative a continuous process with square-integrable values. This implies, that the only nonvanishing term in the above finite sum corresponds to $k = 0$, so that $\mathbf{E}(\nu_t|y_0^t) = D(t)z(t)$ and (15.1.11) is proved.

In order to find the matrix D we put

$$K(t,s) = \mathbf{E}(x_t \otimes z_s) = \mathbf{E}(m_t \otimes z_s),$$

so that

$$m_t = \int_0^t K(t,s)z(s)ds \tag{15.1.13}$$

is the canonical representation of m_t. Then,

$$K(t,s) = \mathbf{E}(x_t \otimes C(x_s - m_s)) = \mathbf{E}(x_t \otimes (x_s - m_s))C^*,\ C = C(s),$$

because $z_s = C(x_s - m_s) + \eta_s$ and η_s is independent of x_t. Now, we obtain from (15.1.11) and (15.1.13) that

$$\frac{\partial}{\partial t}K(t,s) = A(t)K(t,s) + D(t)\delta(t-s).$$

Exercise 15.1.8 *Deduce from the above identity, that* $D(t) = K(t,t)$.

Moreover, it is easy to see (provide details!) that

$$\mathbf{E}(x_t \otimes (x_s - m_s)) = \mathbf{E}((x_t - m_t) \otimes (x_s - m_s)) = R(t,s),$$

where the last identity is the definition. Thus, we deduce that

$$\dot{m} = Am + PC^*z, \tag{15.1.14}$$

where $P(t) = R(t,t)$.

15.1.3 The Riccati equation

Now, we obtain from (15.1.14) and (15.1.3) that

$$\begin{aligned} \dot{m} &= Am + PC^*C(x-m) + PC^*\eta \\ \dot{x} &= Ax + B\xi, \end{aligned} \tag{15.1.15}$$

and, therefore, for $\widetilde{x} = x - m$ we get

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\zeta, \text{ where } \widetilde{A} = A - PC^*C, \widetilde{B} = [B \ -PC^*], \zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{15.1.16}$$

Here, $[B \ -PC^*]$ is a compound matrix made of columns of matrices B and $-PC^*$. The above equation implies, that

$$\dot{P} = \widetilde{A}P + P\widetilde{A}^* + \widetilde{B}\widetilde{B}^*, \tag{15.1.17}$$

where $\widetilde{B}\widetilde{B}^* = BB^* + PC^*CP$. Indeed, the general statement is this:

Suppose, that

$$\dot{x} = Ax + B\xi, \ x(0) = 0,$$

where ξ is a white noise, and $P(t) = \mathbf{E}(x(t) \otimes x(t))$. Then,

$$\dot{P} = AP + PA^* + BB^*. \tag{15.1.18}$$

To prove (15.1.18) we use the Cauchy formula $x(t) = \int_0^t \Phi(t,s)\xi(s)ds$. Then

$$\begin{aligned} &P(t) = \mathbf{E}\left(\int_0^t \Phi(t,u)B(u)\xi(u)du \otimes \int_0^t \Phi(t,v)B(v)\xi(v)dv\right) = \\ &\int_0^t \int_0^t \Phi(t,u)B(u)\mathbf{E}(\xi(u)\otimes\xi(v))B(v)^*\Phi(t,v)^*dudv = \\ &\int_0^t \Phi(t,s)B(s)B(s)^*\Phi(t,s)^*ds, \end{aligned}$$

where the last equality follows from $\mathbf{E}(\xi(u)\otimes\xi(v)) = \delta(u-v)$.
By differentiating $P(t)$ we obtain

$$\begin{aligned} &\dot{P}(t) = B(t)B(t)^* + \int_0^t A(t)\Phi(t,s)B(s)B(s)^*\Phi(t,s)^*ds + \\ &\int_0^t \Phi(t,s)B(s)B(s)^*\Phi(t,s)^*A(t)^*ds = \\ &A(t)P(t) + P(t)A(t)^* + B(t)B(t)^*, \end{aligned}$$

which coincides with (15.1.18). Finally, we get that

$$\dot{P} = AP + PA^* - PC^*CP + BB^*, \tag{15.1.19}$$

which is the celebrated Riccati equation.

15.1.4 The Kalman-Bucy filter

The final equations

$$\begin{aligned} \dot{m} &= (A - PC^*C)m + PC^*y \\ \dot{P} &= AP + PA^* - PC^*CP + BB^* \end{aligned} \tag{15.1.20}$$

determine the classical Kalman – Bucy filter. The equation (15.1.14) is another equivalent form of the first equation (15.1.20).

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