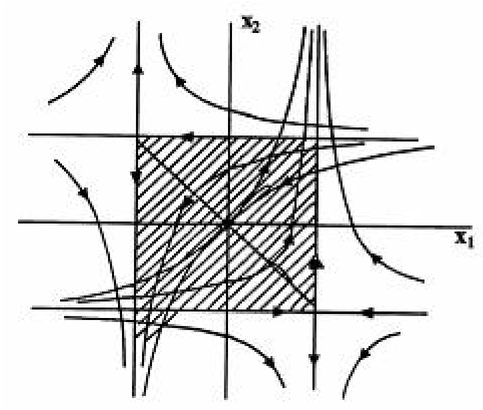
Daniela Marinova

OPTIMAL CONTROL

CLASSICAL APPROACHES AND NEW TRENDS



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OPTIMAL CONTROL
(CLASSICAL APPROACHES AND NEW TRENDS)
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Preface

Beginning in the late 1950s and continuing today, the issues concerning dynamic optimization have received a lot of attention within the framework of control theory. The impact of optimal control is witnessed by the magnitude of the work and the number of results that have been obtained, spanning theoretical aspects as well as applications. The need to make a selection (inside the usually large set of different alternatives which are available when facing a control problem) of a strategy both rational and effective is likely to be one of the most significant motivations for the interest devoted to optimal control.

A further, and not negligible, reason originates from the simplicity and the conceptual clearness of the statement of a standard optimal control problem: indeed it usually requires specifying the following three items:

(a) The equations which constitute the model of the controlled system;

(b) The criterion, referred to as the performance index, according to which the system behavior has to be evaluated;

(c) The set of constraints active on the system state, output, control variables, not yet accounted for by the system model.

The difficulties inherent in points (a) and (c) above are not specific to the optimization context, while the selection of an adequate performance index may constitute a challenging issue. Indeed, the achievement of a certain goal (clearly identified on a qualitative basis only) can often be specified in a variety of forms or by means of an expression which is well defined only as far as its structure is concerned, while the values of a set parameters are on the contrary to be (arbitrary) selected. However, this feature of optimal control problems, which might appear as capable of raising serious difficulties, frequently proves to be expedient, whenever it is suitably exploited by the designer, in achieving a satisfactory trade-off among various, possibly conflicting, and instance through a sequence of rationally performed trials.

The flexibility of optimal control theory together whit the availability of suitable computing instruments has occasionally caused an excess of confidence in its capability to solve (almost all) problems, thus exposing it to severe criticisms and, as a reaction, giving rise to a similarly unjustified belief that it was a formally nice, but essentially useless, mathematical construction. The truth lying somewhere between these two extreme points, the contribution of optimal control theory can be evaluated in a correct way only if an adequate knowledge of its potentialities and limits has been acquired. In this perspective the motivation of the present book is twofold: from one side it aims to supply the basic knowledge of optimal control theory, while from the other side it provides the required background for the understanding of many recent and significant developments in the field (one for all, the control of Hardy spaces) which are undoubtedly and deeply rooted in such a theory.

Three out of the many possible forms of *rendezvous* problems are now briefly presented in order to shed light on some typical aspects of optimal control problems: for this reason it is useless to mention explicitly the set of equations which describe the dynamic behavior of the controlled system and constitute the main, always present, constrain.

Problem 1: The initial state x_0 is given, while the final time t_f when the *rendezvous* takes place is free since it is the performance index J to be minimized. Thus

$$J = \int_{t_0}^{t_f} dt \, .$$

Besides the constraint $x(t_f) = x_b(t_f)$ (which is peculiar to the problem), other requirements can be set forth us, for instance, $u_m \le u(t) \le u_M$ which account for limits on the control actions.

Problem 2: The initial state x_0, the initial time t_0 and the final time t_f are given. The final state is only partially specified (for instance, the final position is given, while the final velocity is free inside a certain set of values) and the performance index aims at evaluating the global control effort (to be minimized) by means of an expression of the kind

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} r_i u_i^2(t) dt, \quad r_i > 0$$

where u_i is the *i*-th component of the control variable *u*. The peculiar constraint of the problem is $x(\tau) = x_b(\tau)$, where the time τ when the *rendezvous* takes place must satisfy the condition $t_0 < \tau < t_f$ and may or may not be specified.

Problem 3: This particular version of the *rendezvous* problem is sometimes referred to as the *interception* problem. The initial state x_0 may or may not be completely specified, while both the initial and final times t_0 and t_f are to be selected under the obvious constraint $t_0 < t_f \le T$. The final state is free and the performance index is as in Problem 2. The peculiar constraint of the problem involves some of the state variables only (the positions), $\xi(\tau) = \xi_b(\tau)$, where the time τ when interception takes place may or may not be given and satisfies the condition $t_0 < \tau < t_f$.

Chapter 1.

Control of systems

1.1 Preliminary

Modern *Control theory* or *Theory of optimal control* started in the 1950s. It was to a large extent of the classical *Calculus of variations*. However, it actually contained much more. In more general sense, *Control theory* can be seen as an extension of the theory of differential equations or of dynamical systems. It is a study of global properties of certain families of dynamical systems.
From the beginning, control theory leans towards application. Some examples in physical systems are stable performance of motors and machinery and optimal guidance of rockets. In management it is optimal exploitation of natural resources. In economics it is optimal investment or production strategies. In biology and medicine these are regulation of physiological functions, fighting against insects and so on.
Any system described by differential equations of the evolution type, can be converted into a control system by adding an input variable representing the action on some *controller* upon the system. The interpretation of this control as a willful action of a person is just one possibility. The added input can also be called "noise" and represent many factors about which we may not have any influence nor even knowledge.

1.2 Mathematical Modeling

When attempting to study the behavior of certain systems, it is convenient to consider the ideal case of an ***isolated system***, i.e. a number of interesting elements, which do not have any interaction with respect of the world. In such *isolated systems* the conditions are simpler, and therefore easier to study.
We may consider systems, which are isolated except for some well defined actions affecting them. We think of this outside action, also called ***input*** or *control* as the result of decisions of a *controller*. The important information we need is a rile, within the description of the system, of which inputs are possible and which are not. The possible inputs will then be called *admissible*.
The big difference between systems without and with inputs lies in the type of problems, which are meaningful to be posed for them. For systems without inputs, the basic problem is to predict the future behavior. For this purpose the differential equations are exactly tailored. But the prediction of future evolution is not the only meaningful problem to be posed. The whole field of engineering and technology deals with the *inverse* problem: given a desired future evolution, how should we construct the system?
For systems with inputs two of the basic questions are: (a) given the initial conditions, which are the "states" of the system, which we can reach by choosing suitable inputs? and (b) which are the *best* inputs to be used in some well prescribed sense?
The mathematical description of a system is given by a differential equation of the type

$$\dot{x} = f(t, x), \tag{1.2.1}$$

where t is the time and $x = (x_1, x_2, ..., x_n)^T$ is the state. The function $f : R \times R^n \to R^n$, representing the laws governing the evolution of the system, is assumed to be known. Together with the initial conditions

$$x(t_0) = x_0, \tag{1.2.2}$$

(1.2.1) determines the solution $x(t)$ uniquely. If the function *f* in (1.2.1) does not depend on *t*, i.e. of the form

$$\dot{x} = f(x), \tag{1.2.3}$$

it means that the system is *invariant in time* and is called ***autonomous***. For an autonomous system, if $x(t)$ is a solution, then $x(t - t_0)$ is also a solution for any t_0. Autonomous systems are also called ***dynamical systems***. They are geometrically appealing, since the trajectories are fixed curves in $\aleph$ -space.
Any non-autonomous system can be transformed into an autonomous one by increasing the dimension of the state space R^n in one, practically transforming the time t into an additional coordinate of the state space. Hence, most properties of autonomous control systems can be extended, in some sense, to non-autonomous ones. But doing so, in many cases the corresponding properties loose their interest.
A ***control system*** will be described by a differential system

$$\dot{x} = f(t, x, u), \tag{1.2.4}$$

where t is the time (independent variable), $x \in R^n$ is the state of the system and $u = (u_1, u_2, ..., u_m)^T \in R^m$ is the *control*. The control is assumed to be an arbitrary function $u(t)$, but some restrictions must be imposed. It must be *measurable* function. Another restriction of the control which appears in many applications is the requirement that the values of $u(t)$ belong to a specified set U

$$u(t) \in U \qquad (U \in R^m) \tag{1.2.5}$$

An ***admissible control*** is therefore a measurable function $u(t)$ satisfying (1.2.5). The set U can be fixed or depending on *t* and/or *x*. For each admissible control $u(t)$, (1.2.4) is a differential equation

$$\dot{x} = f(t, x, u(t)).$$

Its solution $x(t)$ (expected to exist and be unique) is then also called ***admissible solution*** (also *trajectory* or *orbit*).
For an autonomous control system $f(t, x, u)$ does not contain *t* and the constraint $u \in U$ is also independent of *t*.

$$\dot{x} = f(x, u), \qquad u \in U, \tag{1.2.6}$$

The pairs of admissible control and admissible solution are time invariant. When the "modern control theory" developed, it was mostly as a "theory of optimal control". The techniques involved were from the theory of differential equations, but the problem setting was usually an optimization of a given functional. It was, therefore, a somewhat particular case within the much older "classical calculus of variations".

The classical calculus of variations extends the theory of maxima and minima from calculus to functional analysis, where the unknown is not a value of *x*, but a whole function $x(t)$.

In classical theory, the unknown is consistently assumed to be an *interior point* of some domain in this function space. In many important problems this is certainly not the case. The customary linear feedback controls did not satisfy all the requirements of the emerging applications. Discontinuous feedback controls were quite efficient but not fit into any general theory. Pontryagin succeeded in developing a theory of control optimization which, contrary to the classical calculus of variations, could take care of discontinuous controls $u(t)$ and *unilateral constraints*.

1.3 General properties of control systems

1.3.1 Definitions

We will refer mainly to the continuous case for processes described by differential equations

$$\dot{x} = f(t, x, u), \qquad u(t) \in U, \tag{1.3.1}$$

where $t \in I \subset R$ is thought as the time, $x \in \aleph \subset R^n$ is the state and $u \in U \subset R^m$ is the control. The state *x* and the control *u* are assumed to be functions of *t*. This control system is therefore determined by the function *f* and the control set *U*. The function $f : I \times R^n \times U \to R^n$ is assumed continuous in (x, u), Lipschiz in *x*, measurable in *t*. Furthermore, it is assumed that for all $u \in U$ and *x* in any given compact subset of R^n, there exists a majorizing integrable function $m(t)$ such that

$$\| f(t, x, u) \| \le m(t).$$

Under these conditions, given t_0, x_0 as initial conditions

$$x(t_0) = x_0, \tag{1.3.2}$$

and given a measurable function $u(t)$ with values in *U*, defined in some interval $I = [t_0, t_1]$,the well known Caratheodory conditions guarantee the existence and uniqueness of a solution $x(t)$ of the initial value problem (1.3.1),(1.3.2), at least in some interval $[t_0, t^*)$.

The admissible control $u(t)$, defined on a suitable *t*-interval *I*, is an integrable function mapping *I* into the control set *U*.

For each particular choice of the admissible control $u(t)$, the problem reduces to the integration of the differential equation (3.1) (where $u(t)$ is then known), hence, control theory could be expected not to present any new problems. But this only refers to finding particular solutions $x(t)$, while a great variety of other problems can also be posed referring to the set of all solutions. Let us see a simple example.

Example 1.3.1

Let the state space $\aleph$ be the real line and consider the control system defined by

$$\dot{x} = u, \qquad |u(t)| \le 1. \tag{1.3.3}$$

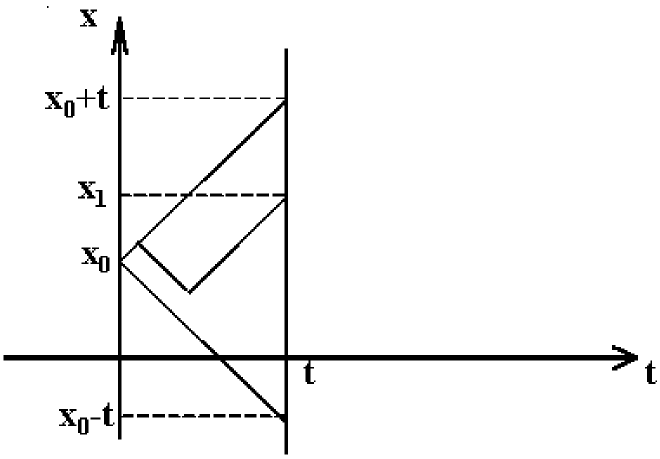
With the initial condition

$$x(0) = x_0$$

the solution is

$$x(t) = x_0 + \int_0^t u(s)ds \, . \qquad (1.3.4)$$

Now we can start posing problems. A basic question: what is the set of points $x(t)$ obtained for all possible choices of admissible controls $u(s)$, for a given end-time *t*? This set will be called ***Attainable set at t, starting from*** x_0 ***at time*** t_0, and denoted by $A(t, t_0, x_0)$.
It is immediate that, with a maximum velocity equal to 1 in either direction, the attainable set is the interval between the points $x_0 \pm t$. These two end-points can be attained in only one



way: using (3.4), $u(s) = 1$ or $u(s) = -1$ all the time. For all the points like x_1 in between, the control $u(s)$ satisfying (3.3) and accomplishing $x(t) = x_1$ is not uniquely determined.

The class of control systems which are linear in (x, u), i.e. are of the form

$$\dot{x} = A(t)x + B(t)u + c(t)$$

with A and B suitable matrices and *c* a vector, are very important. They can model many applications and at the same time are easy to study analytically. In the most cases the term $c(t)$ is set equal to zero, since this is the case in most applications. Hence, we will consider the ***linear control system***

$$\dot{x} = A(t)x + B(t)u \qquad (1.3.5)$$

with initial conditions

$$x(t_0) = x_0 \, . \qquad (1.3.6)$$

The solution of initial-value problem (1.3.5),(1.3.6) is given by the "variations of parameters" formula

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, s)B(s)u(s)ds \, . \qquad (1.3.7)$$

Here, $\Phi(t, t_0)$ is the *fundamental matrix* of the homogeneous system

$$\dot{x} = A(t)x \, .$$

This fundamental matrix satisfies the matrix differential equation

$$\frac{\partial\Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0) \tag{1.3.8}$$

with the initial condition

$$\Phi(t_0,t_0) = I\ , \tag{1.3.9}$$

where *I* is the identity matrix. It also satisfies the inevitability relation

$$\Phi^{-1}(t,t_0) = \Phi(t_0,t)\ . \tag{1.3.10}$$

Setting $u(t)=0$, i.e. considering the *uncontrolled* or *free* system, the solution $x(t)$ is given by the first term of the right side of (1.3.7). On the other hand, if we set $x_0 = 0$, the solution will be given by the last term of (1.3.7). This gives the following *superposition principle*:

The solution of the linear control system (1.3.5) is the sum of the solution of the homogeneous system with the given initial condition (1.3.6) plus the solution of the non-homogeneous system (1.3.5) with initial condition zero.

Consider the system (1.3.5)-(1.3.6) with $A(t)$, $B(t)$, U and t, t_0 given. Then $\Phi(t,t_0)$ is determined as solution of (1.3.8)-(1.3.10). If we think of inserting in (1.3.7) all admissible controls $u(s)$, we will obtain all attainable points $x(t)$. Hence, the ***attainable set of the linear control system*** is

$$A(t,t_0,x_0) = \Phi(t,t_0)x_0 + \{\int_{t_0}^{t} \Phi(t,s)B(s)u(s) \,\big|\; u(\cdot) \text{ admissible } \} \tag{1.3.11}$$

the vector-sum of the position of the uncontrolled motion (for $u=0$) starting at x_0 plus the attainable set from the origin.

1.3.2 Attainability relation

Consider an autonomous control system

$$\dot{x} = f(x,u)\ , \qquad u \in U \tag{1.3.12}$$

satisfying the Caratheodory conditions.

Given two points x_0, x_1 in $\aleph$ and two numbers t_0, t_1 in *R*, $t_0 < t_1$, we will say that x_1 ***is attainable at time*** t_1 ***starting from*** x_0 ***at time*** t_0, if there exists an admissible control $u(t)$ defined on $[t_0,t_1]$ with corresponding trajectory $x(t)$ satisfying $x(t_0)=x_0$, $x(t_1)=x_1$. We also say that x_1 is forwards ***reachable*** from x_0.

Notice, both words "attainable" and "reachable" are sometimes used interchangeably, but here we will in general use "attainability" when the end-time is specified and "reachability" when any end-time may apply.

Since for autonomous systems the attainability will depends only on x_0, x_1 and on the difference $t_1 - t_0$, we may refer mostly to $t_0 = 0$ stating that x_1 is attainable from x_0 at t_1.

The following properties of the attainability relation are immediate.

- For t_1, t_2 >0, if *y* is attainable from *x* at time t_1, and *z* is attainable from *y* at time t_2, then *z* is attainable from *x* at time $t_1 + t_2$.

- If *y* is reachable from *x* and *z* is reachable from *y*, then *z* is reachable from *x*.
- Semigroup property of the attainability function: If t_1, t_2 >0, then

$$A(t_1 + t_2, x) = \bigcup_{y \in A(t_1, x)} A(t_2, y) .$$

A necessary and sufficient condition for the existence of a ***periodic solution*** through the point *x* is that *x* be attainable from itself at some positive time *t*, or (equivalently) that *x* be reachable from itself. Consider a control system (1.3.12) and a point $x_0 \in \aleph$. The set of all points $x \in \aleph$ such that *x* is attainable from x_0 and conversely x_0 is attainable from *x* is called ***holding set from*** x_0 and denoted by $\mathrm{H}(x_0)$:

The intuitive meaning of a holding set is quite obvious: it is a subset of the state space such that "we can go" (if "we wish") from x_0 to any other point $x \in \mathrm{H}(x_0)$ and then "come back" to x_0. Since this "going and coming back" will take some finite positive time, we could then repeat this (if "we wish") indefinitely and stay within H forever. But once we left this holding set $\mathrm{H}(x_0)$, we can never come back to x_0.

The subset of $\aleph$ of all points *x* with $\mathrm{H}(x) = 0$ is called the ***transient set*** and denoted by T. *x* belongs to the transient set T is there is no admissible solution starting at *x* and returning to *x* after some positive time *t*.

Some properties of the holding sets are.

- If both *y* and *z* belong to $\mathrm{H}(x)$, then $y \in \mathrm{H}(z)$ and $z \in \mathrm{H}(y)$.
- If $\mathrm{H}(x) \neq 0$, then $x \in \mathrm{H}(x)$.
- If $\mathrm{H}(x) \neq 0$, then there is a periodic solution through *x*.

The control system (1.3.12) determines a decomposition of the state space $\aleph$ into: (i) the transient set and (ii) all the holding sets, equivalent classes of the mutual reachability.

1.3.3 Linear systems

Before showing an example for the holding sets we will tell some words about the autonomous linear control systems. These are of the form

$$\dot{x} = Ax + Bu , \qquad u \in U . \tag{1.3.14}$$

The matrices *A* and *B* as well as the control set *U* must be constant. For constant *A*, the fundamental matrix of the homogeneous system $\dot{x} = Ax$ is

$$\Phi(t, t_0) = \exp(A(t - t_0)) ,$$

where

$$\exp(At) = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \dots$$

If the control is constant, the system can be solved elegantly. For $u = 0$ in (1.3.14) the solution of the differential equation

$$\dot{x} = Ax \tag{1.3.15}$$

is

$$x(t) = \Phi(t, t_0) x_0 .$$

If u=const. and the matrix A is nonsingular the equation (3.14) can be rewritten with the change of variables

$$\xi = x + A^{-1}Bu\,,$$

obtaining

$$\dot{\xi} = \dot{x} = Ax + Bu = A(x + A^{-1}Bu) = A\xi\,,$$

which is the same equation as (1.3.15). Hence, the solution is $\xi(t) = \Phi(t,t_0)\xi_0$ in the new coordinates, and the geometric graph of the trajectories is the same as for the homogeneous system (1.3.15), but translated to the new origin $\xi_0 = 0$, which is $x_0 = -A^{-1}Bu$. If A is singular but there exists a matrix C such that AC=B, the transformation of coordinates $\xi = x + Cu$ will do the same as before. If such matrix C does not exist, then the trajectories of the system controlled with u=const. are not the translates of the trajectories of the homogeneous system.

Example 1.3.2 (*Stable node*)
Consider the control system

$$\dot{x}_1 = -x_1 + u\,, \qquad \dot{x}_2 = -2x_2 + u\,, \qquad |u| \le 1\,. \tag{1.3.16}$$

Here the homogeneous system

$$\dot{x}_1 = -x_1\,, \qquad \dot{x}_2 = -2x_2$$

has a matrix A with two distinct real negative eigenvalues, hence, the origin is a stable node and the solutions are given by

$$x_1 = x_1(0)e^{-t}\,, \qquad x_2 = x_2(0)e^{-2t}\,.$$

Hence

$$x_2 = cx_1^2$$

with c being the constant adjusting the initial condition. The trajectories are half-parabolas converting to the origin.
We absorb the constant controls into the coordinates, by rewriting the system equations

$$\text{for } u = 1 \text{ as} \qquad \dot{x}_1 = -(x_1 - 1)\,, \qquad \dot{x}_2 = -2(x_2 - \frac{1}{2})\,;$$

$$\text{for } u = -1 \text{ as} \qquad \dot{x}_1 = -(x_1 + 1)\,, \qquad \dot{x}_2 = -2(x_2 + \frac{1}{2})\,.$$

The parabolas convert to the points of attraction $(1,\frac{1}{2})$ for $u = 1$ and $(-1,-\frac{1}{2})$ for $u = -1$. Starting at any initial point, we can choose to move along each of these two families of parabolas. We may switch at any time from one to the other family.
Now we will find out which points x are reachable from a given x_0. Setting $u = 1$ for a while, the point $x(t)$ will drift asymptotically towards the center of attraction $(1,\frac{1}{2})$. When it is near this point we may then switch to $u = -1$, so that the moving point $x(t)$ will start moving

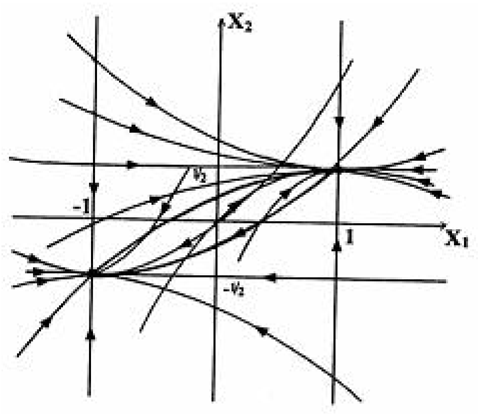
asymptotically towards $(-1,-\frac{1}{2})$. And when it is near this point, we may of course switch back again and continue repeating this as many times as we wish.
The conclusion is that once the point $x(t)$ is in the lunette limited by the two points of attraction and the corresponding arcs of parabola, it can never get out of it (it can not even reach its boundary in finite time). On the other hand, within this lunette it can go from any point to any other point. Therefore, the interior of the lunette is a holding set.

It is important to observe that every vector-differential equation of the form

$$\dot{x} = Ax$$

can be transformed by a linear change of coordinates into a ***Jordan canonical form*** with the new matrix *A* diagonal or made of special blocks. In the case some of the eigenvalues of *A* are complex, one should use the real canonical form.
Introducing the same change of coordinates to the control system

$$\dot{x} = Ax + Bu\,,$$



matrix *A* will change into its canonical form. The matrix *B* will change into whatever new matrix comes out, but the discussion will be easier since *A* is canonical.

1.3.4 Controllability

If for a certain control system a point x_1 is not reachable from another point x_0, this can be due to one of the following reasons:

- the admissible controls are not strong enough to overcome the other "forces" of the system;
- all admissible controls act in directions such that, no matter how strong is the control, $(t, x(t))$ stays on some lower dimensional surface in R^{n+1}. For example consider the system

$$\dot{x}_1 = -x_1 + x_2 \qquad \dot{x}_2 = -x_1 - x_2 + u \qquad \dot{x}_3 = x_3 \text{ with } |u| \le 1.$$

It decomposes into two uncoupled subsystems: the two first equations and the last one. No matter how we choose the control $u(t)$, the solution of the third equation

$$x_3 = x_3(0)e^t$$

will be unaffected. This means that we cannot control x_3 and this system will be called ***not controllable***.

Consider the control system with unbounded controls

$$\dot{x} = Ax + Bu, \qquad x \in R^n, \quad u \in R^m \tag{1.3.17}$$

Let us find the attainable set from origin. With $x_0 = 0$ the solution is

$$x(t) = \int_0^t \Phi(t,s)Bu(s)ds$$

and the attainable set, a linear subspace of R^n, is

$$A(t,0) = \{\int_0^t \Phi(t,s)Bu(s) \mid u(\cdot) \text{ attainable}\}.$$

If u_1 and u_2 are admissible controls with respective trajectories x_1 and x_2 starting at the origin, then $k_1u_1 + k_2u_2$ is also an admissible control for any real k_1, k_2, and by linearity, the corresponding trajectory will be $k_1x_1(t) + k_2x_2(t)$. The conclusion is that if x_1 and x_2 are attainable from the origin at time *t*, then any linear combination $k_1x_1 + k_2x_2$ is also attainable at *t*.

The linear control system (1.3.17) is ***controllable*** if for every $t > 0$ the attainable set is

$$A(t,0) = R^n. \tag{1.3.18}$$

This immediately implies that also $A(t, x_0) = R^n$ for any x_0.

Denote by $M(t)$ the matrix

$$M(t) = \int_0^t \Phi(t-s)BB^T\Phi^T(t-s)ds. \tag{1.3.19}$$

Theorem: A necessary and sufficient condition for system (1.3.17) to be controllable, is that the matrix *M* defined by (1.3.19) be positive definite, i.e. of full rank.

An equivalent theorem is the following.

Theorem: A necessary and sufficient condition for the controllability of the system (1.3.17) is that the controllability matrix

$$[B, AB, A^2B, ..., A^{n-1}B] \tag{1.3.20}$$

has maximum rank (i.e. rank *n*).

1.4 Optimal control and related results

1.4.1 Optimal control

Consider the control system

$$\dot{x} = f(t, x, u), \tag{1.4.1}$$

with a constant in time constraint

$$u(t) \in U \,. \qquad (1.4.2)$$

There may be an initial condition

$$x(t_0) = x_0$$

and possible end-condition

$$x(t_f) = x_f \,.$$

In order to optimize something, we define an *objective functional* (also called *cost* or *gain functional*) of type

$$J = g(t_f, t_f) + \int_{t_0}^{t_f} f_0(s, x(s), u(s))ds \qquad (1.4.3)$$

evaluated along the solution $x(t)$ corresponding to the control $u(t)$. The problem is to find (if it exists) the ***optimal control*** $u^*(t)$ generating the ***optimal trajectory*** $x^*(t)$ such that the corresponding cost J^* is minimum (or maximum). This is called the *problem of Bolza*.
If the term $g(t_f, t_f)$ is absent and (4.3) is of the form

$$J = \int_{t_0}^{t_f} f_0(s, x(s), u(s))ds \,, \qquad (1.4.4)$$

it is called the *problem of Lagrange*. When the integral is absent, i.e.

$$J = g(t_f, t_f) \,, \qquad (1.4.5)$$

it is called the *problem of Mayer*. It can be seen that these problems are equivalent, in the sense that each of them can be rewritten as any other type of them.
Quite different is the ***time optimal problem***.

$$x = f(t, x, u) \,, \qquad u(t) \in U \quad \text{given,}$$

$$x_0 = 0 \,, \qquad x(t_f) = x_f \quad \text{given, minimize } t_f \,.$$

Here $$J = \int_0^{t_f} 1dt \,.$$

It is a Lagrange problem, but with the end-time not prescribed. We wish to drive, using an admissible control, $x(t)$ from the origin to the given target point x_f in minimal time.

Once the optimal control problem had been reformulated with inequality constraints on the admissible controls, necessary conditions for optimality had to be found to replace the classical ones from the calculus of variations. This has achieved by Pontryagin with what is called the ***maximum principle***. Here we state it only for a Mayer problem with linear cost functional.
Consider the problem defined by

$$\dot{x} = f(x, u,) \,, \; u(t) \in U \,, \; x(0) = x_0 \,, \text{ maximize } \; J = \eta x(t_f) \,, \; (4.6)$$

where $x \in R^n$, $U \subset R^m$ compact, $f(x,u)$ continuously differentiable, with η and t_f given. Assume that the admissible optimal control $u^*(t)$ with corresponding optimal trajectory

$x^*(t)$ achieve the maximum of *J*. Then there exists a non-zero *n*-dimensional *adjoined vector* function $p(t)$ satisfying the *adjoined differential equation*

$$\dot{p} = -p\left[\frac{\partial f}{\partial x}\right]_{x=x^*(t),u=u^*(t)} \qquad (1.4.7)$$

where $\left[\frac{\partial f}{\partial x}\right]_{x=x^*(t),u=u^*(t)}$ is the jacobian of partial derivatives of the components of $f(x,u)$ with respect of the components of *x*, evaluated at the optimal solution $x = x^*(t)$, $u = u^*(t)$, such that for almost all *t* in the interval $[0, t_f]$

$$p(t).f(x^*(t),u^*(t)) = \max\{p(t).f(x^*(t),u(t)) \mid u \in U\}. \qquad (1.4.8)$$

In addition, $p(t)$ should satisfy the end-condition

$$p(t_f) = \eta. \qquad (1.4.9)$$

Condition (1.4.7) can be interpreted as follows. Once $p(t)$ is known, the optimal control $u^*(t)$ should be chosen at each instant *t* as to maximize the scalar product $p(t).f(x^*(t),u)$, hence the term "maximum principle".
For linear control systems of the type

$$\dot{x} = Ax + Bu, \qquad u \in U$$

the maximum principle applies in a particularly simple way. Here $\frac{\partial f}{\partial x} = A$ and the adjoined equation (1.4.7) becomes

$$\dot{p} = -pA.$$

It does not depend on either $u(t)$ nor $x(t)$ and can be integrated independently of the optimization criterion. Once the general solution of the adjoined system is obtained, each value of η in (1.4.9) will determine a particular solution $p(t)$. This is especially well adapted to determine the attainable set of a linear control system at a given final time t_f.

1.4.2 The existence of optimal control

We will continue with an example. Consider the optimal control problem

$$\dot{x}_1 = \frac{|u|}{1+|x_2|}, \qquad \dot{x}_2 = \frac{u}{1+|x_2|}, \qquad u \in [-1,+1]. \qquad (1.4.10)$$

Starting at the origin, $x_1(0) = x_2(0) = 0$, we wish to minimize

$$J = x_1(1).$$

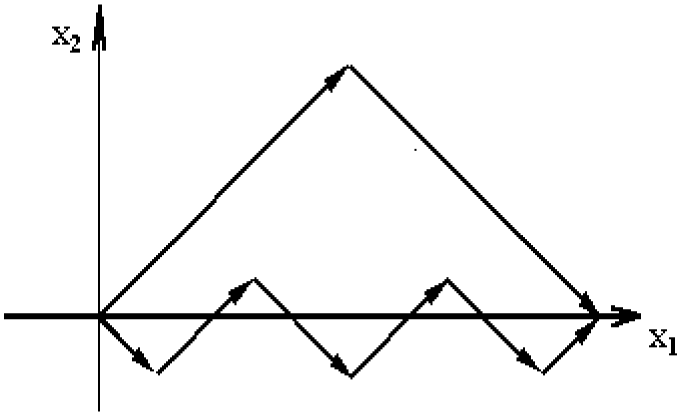
For $u = 0$, $\dot{x}_1 = \dot{x}_2 = 0$ is not desirable. For all other values of *u*,

$$\frac{\dot{x}_2}{\dot{x}_1} = \pm 1,$$

hence, the trajectories are of the zig-zag type of slop ± 1, as seen in figure.. Since

$$x_1(1) = \int_0^1 \frac{|u|}{1+|x_2|} ds \quad \rightarrow \quad \text{to be maximized.}$$

We wish to keep $u = \pm 1$ and also keep $|x_2|$ as small as possible. The best would be to keep $x_2 = 0$, but this is not possible since the trajectories have to have constantly slope ± 1.
Therefore, to make *J* large, we should use $u(t) = \pm 1$ and switch quite often, in order to obtain



a zig-zag with $x_2(t)$ kept near zero. As more zig=zag's, as better the *J*. The limit would be to have $\dot{x} = 1$ and $x_2(t) = 0$ all the time, but this is impossible because it does not correspond to any admissible control. The conclusion is that, for any positive ε, values of $J > 1 - \varepsilon$ are possible to obtain, but $J = 1$ is impossible. Hence an optimal control does not exist. The attainable set $A(1,0)$ is not closed, since it does not contain the point (1,0). More precisely, if we would have assumptions guaranteeing that $A(1,0)$ is compact set, then we could argue that there exists an optimal control and an optimal trajectory leading to this point.
Since compact in R^n means closed and bounded, and the boundness of the attainable set usually follows from an upper bound of the admissible velicities $\|\dot{x}\|$, the crucial point is therefore to give some sufficient conditions to insure that the attainable set is closed.

Theorem 1.4.1 Consider the control system

$$\dot{x} = f(t,x,u), \quad u \in U \subset R^m$$

and assume that *f* satisfies the conditions:

- $f : I \times R^n \times U \rightarrow R^n$ is continuous in (x,u) and integrable with respect to *t* for every (x,u), $x \in \aleph$, $u \in U$;
- *f* is lipschitz in *x*, i.e. $\|f(t, x_{(1)}, u) - f(t, x_{(2)}, u)\| \leq K \|x_{(1)} - x_{(2)}\|$;
- for all $u \in U$, $\|f(t,x,u)\| \leq \mu(t) \cdot \psi(\|x\|)$,where $\mu(t)$ is integrable in each finite interval and $\psi(\|x\|)$ is bounded in any bounded region of R^n;
- the admissibility condition: $u(t)$ is measurable with values in *U*;
- the convexity condition: $f(t,x,U) = \{ f(t,x,u) \, | u \in U \}$ is convex.

Then the attainable set $A(t, x_0)$ is bounded and closed, hence compact. The last convexity assumption is violated in the example.

Theorem 1.4.2 The attainable set of a linear control system, assuming *U* convex, is convex.

Theorem 1.4.3 Assume two control systems with the same matrices *A*, *B* but different control sets U_1 and U_2. If both control sets have the same convex hull:

$$co(U_1) = co(U_2) .$$

Then the attainable sets are the same

$$A_1(t,0) = A_2(t,0) .$$

This result is remarkable. From it follows that in all the linear systems where the control set is the interval $U = [-1,+1]$, this could be replaced by the set of just the two end-points $U = \{-1;+1\}$.
In the more general case of *U* being a convex polygon, it means that any point attainable with controls taking values in this polygon, can also be attained with controls taking values only on the vertices of *U*. This has been called *bang-bang* principle.

1.4.3. Invariant sets

Invariant sets are well known in the theory of dynamical systems. They also play an important role for control systems. The fundamental difference in both cases is that in dynamical systems the starting point $x(0)$ determines uniquely the whole past and future trajectory $x(t)$, while for control systems there is a whole set of trajectories passing through a given $x(0)$. This immediately induces two ways to apply any property to control systems. We denote these two possibilities by *strong*, if the property applies to all admissible trajectories, and by *weak* if the property applies to some trajectories (at least one).
We refer to autonomous control system of the type

$$\dot{x} = f(x,u), \qquad u \in U , \tag{1.4.11}$$

where *f* satisfies the conditions of the Theorem 1.4.1.

Definition 1.4.1 Let *S* be a subset of the state space $\aleph$. The set is said to be ***strongly invariant*** for a given control system, if for every $x_0 \in S$, all admissible trajectories $x(t)$ through $x(0) = x_0$ remain in *S* for all future,

$$x(0) \in S \quad \text{implies} \quad x(t) \in S \quad \text{for all } t \geq 0 \tag{1.4.12}$$

Definition 1.4.2 Let *S* be a subset of the state space $\aleph$. The set is said to be ***weakly invariant*** for a given control system, if for every $x_0 \in S$ there exists an admissible trajectory $x(t)$ with $x(0) = x_0$, remaining in *S* for all future times.

In other words: starting at any x_0 in *S*, we can, by choosing a suitable admissible control, make $x(t)$ remain in *S* forever. For control systems, for which the attainable set $A(t, x_0)$ is compact, and for closed sets *S*, this definition is equivalent to ask that the intersection $A(t, x_0) \cap S \neq 0$ for every $t \geq 0$.
The case that the set *S* in the above definitions is a single point is of particular importance. ***Strongly invariant points*** appear only rarely in applications. They can be called *fixed points*.

The ***weakly invariant points*** are usually called *rest points*. For the system of type (1.4.11) they are the points x_r such that the set

$$\{u \in U \mid f(x_r, u) = 0\} \tag{1.4.13}$$

is not empty. If that is the case, we just need to choose one such solution of (1.4.13) as constant control $u(t)$, getting $\dot{x} = 0$ and remaining at the rest point forever.
To find the location of the rest points is in many cases the first step for analyzing the behavior of a control system.

1.4.4 Stability of invariant sets

The stability properties of invariant sets are also well known from the theory of dynamical systems. Again, they can be applied to control systems in both a strong and a weak form.
For a set *S*, a *neighborhood V* of *S* can be understood in a topological sense that

$$\text{closure of } S \subset \text{interior of } V.$$

Definition 1.4.3 Let *S* be a bounded strongly invariant set. Then *S* is called ***strongly stable*** if, given any neighborhood *V* of *S*, there exists a neighborhood *W* of *S* such that if x_0 is in *W*, then the forward reachable set $R^+(x_0) \subset V$.

It follows that, under these conditions, the attainable set $A(t, x_0) \subset V$ for every $t > 0$. In less precise terms: starting sufficiently near *S*, any trajectory will always remain near *S*.

Definition 1.4.4 Let *S* be a bounded weakly invariant set. Then *S* is called ***weakly stable*** if, given any neighborhood *V* of *S*, there exists a neighborhood *W* of *S* such that if x_0 is in *W*, then there exists an admissible trajectory $x(t)$ though $x(0) = x_0$, remaining in *V* for all future times $t > 0$.

In less precise terms: starting at any point sufficiently near *S*, we can remain (if we wish) near *S* forever.

Definition 1.4.5 Let *S* be a bounded strongly invariant set. Then *S* is called ***strongly asymptotically stable*** if it is strongly stable and there exists a neighborhood *W* of *S* such that for any admissible trajectory $x(t)$ starting at $x(0) = x_0$,

$$\lim_{t \to \infty} \text{distance } (x(t), S) = 0 .$$

Definition 1.4.6 Let *S* be a bounded weakly invariant set. Then *S* is called ***weakly asymptotically stable*** if it is weakly stable and there exists a neighborhood *W* of *S* such that for any $x_0 \in W$, there exists an admissible trajectory $x(t)$ starting at $x(0) = x_0$, with

$$\lim_{t \to \infty} \text{distance } (x(t), S) = 0 .$$

Definition 1.4.7 Let *S* be a bounded strongly invariant set. Then *S* is called ***strongly finite stable*** if it is strongly stable and there exists a neighborhood *W* of *S* such that for any admissible trajectory $x(t)$ starting at $x(0) = x_0$,

$$x(t) \in S \text{ for some finite } t > 0.$$

Definition 1.4.8 Let *S* be a bounded weakly invariant set. Then *S* is called ***weakly finite stable*** if it is weakly stable and there exists a neighborhood *W* of *S* such that for any $x_0 \in W$, there exists an admissible trajectory $x(t)$ starting at $x(0) = x_0$, such that

$$x(t) \in S \text{ for some finite } t > 0.$$

The finite type of stability does not occur in the theory of dynamical systems, since it is ruled out by the uniqueness of a trajectory through a given invariant point x_0. Here, in control systems, it is possible and indicates an even higher degree of stability than the asymptotic stability.

1.4.5 Attractors and repellers

We consider control system of the type (1.4.11).

Definition 1.4.9 A bounded set *S* is called a ***strong attractor***, if it has a neighborhood *V* such that $x_0 \in V$ implies

$$\lim_{t \to \infty} \text{distance } (x(t), S) = 0$$

for every admissible trajectory $x(t)$ starting at $x(0) = x_0$.

Definition 1.4.10 A bounded set *S* is called a ***weak attractor***, if it has a neighborhood *V* such that $x_0 \in V$ implies

$$\lim_{t \to \infty} \text{distance } (x(t), S) = 0$$

for some admissible trajectory $x(t)$ starting at $x(0) = x_0$.

An even stronger version of attraction may require the trajectories not only to approach *S*, but actually to enter this set. We call this ***absorption***.

Definition 1.4.11 A bounded set *S* is called a ***strong repeller***, if it has two neighborhoods *V* and *W*, such that $x_0 \in V$ but not in *S* implies that for every admissible trajectory $x(t)$ starting at $x(0) = x_0$ there is a $t^* > 0$ such that

$$x(t) \notin S \qquad \text{for all } t > 0$$

and

$$x(t) \notin W \qquad \text{for all } t > t^*.$$

Definition 1.4.12 A bounded set *S* is called a ***weak repeller***, if it has two neighborhoods *V* and *W*, such that $x_0 \in V$ but not in *S* implies that there exists an admissible trajectory $x(t)$ starting at $x(0) = x_0$ and a $t^* > 0$ such that

$$x(t) \notin S \qquad \text{for all } t > 0$$

and

$$x(t) \notin W \qquad \text{for all } t > t^*.$$

1.4.6 Attracting, repelling and saddle holding sets

In some way, holding sets are for control systems what critical points are for dynamical systems. We can therefore expect them to also show behavior of attractors, repellers and saddles. We can strengthen the attracting property to the absorbing one. Let us look about the invariance and stability properties of the holding sets.

Proposition 1.4.1 Every holding set is forward weakly invariant.

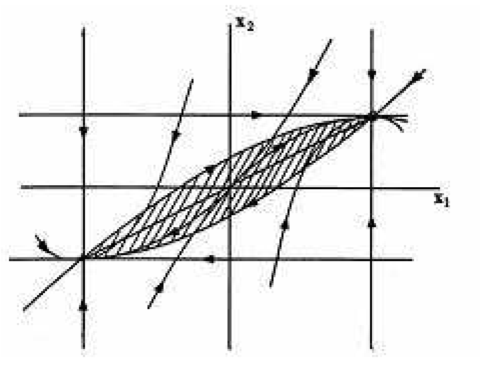
Proposition 1.4.2 If a holding set is weakly absorbing, then it is forwards strongly invariant.

Repelling holding sets are not exactly the opposite of attracting ones. They are unstable, but they do not seem to have otherwise interesting properties.
The typical saddle-behavior of critical points of dynamical systems consists in the existence of some trajectories tending to the saddle point and some going away from it. Similar behavior for control systems can be defined in various ways.

Example 1.4.1 (*Strongly attracting holding set*)
Let us retake the example

$$\dot{x}_1 = -x_1 + u, \qquad \dot{x}_2 = -2x_2 + u, \quad |u| \le 1 \tag{1.4.14}$$

The extreme rest points $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$, and the whole set of rest points is the line segment with these end-points. The lunette determined by the trajectories going from each of the extreme rest points to the other, is a forward invariant set. The holding set from the origin is the interior of this lunette. Besides this, there are actually only two more holding sets: the two extreme rest points, $x(t)$ can stay indefinitely, but once it goes away it never can come back.

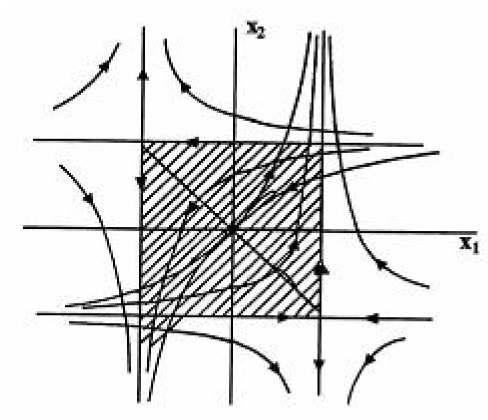


The best way to imagine this holding set being generating is as follows. Any point reachable from the origin can be reached in a time optimal way with *bang-bang* control $u(t) = \pm 1$ with at most one switching point. Hence we can start at the origin with either u=1 or u=-1, and, after an arbitrary time, switch to the opposite control and keep it until the end. This holding set is strongly attracting. But it is not strongly absorbing, since there are nearby trajectories, for example the two extreme rest points, never entering the holding set.

Example 1.4.2 (*Saddle holding set*)
Let us take the system

$$\dot{x}_1 = -x_1 + u\,, \qquad \dot{x}_2 = x_2 + u\,, \qquad |u| \le 1\,,$$

with eigenvalues 1 and -1.Without control there is a saddle point at the origin. For *u*=+1 there is a rest point at (1,-1), and for *u*=-1 there is a rest point at (-1,1). The points of the line segment of these end-points are all rest points for values of *u* inbetween 1 and -1. The horizontal component of the motion is an attraction of the origin or corresponding rest point, while the vertical component of the motion is a repulsion by the origin or rest point.



Observing how the trajectories of this system are described, and in particular their relation with the extreme trajectories (for $u = \pm 1$), it is easy to recognize that the interior of the square of vertices $(\pm 1, \pm 1)$ is the holding set from the origin. This holding set is a sink-source saddle: trajectories enter on the two sides and leave on the top and bottom.

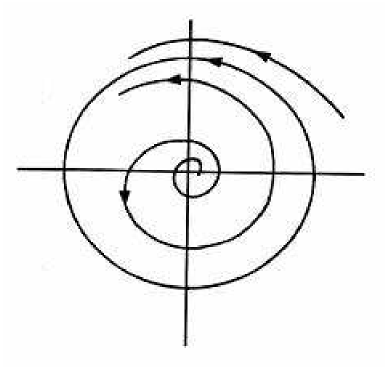
1.4.7 Periodic orbits

The problem of periodic solutions for control systems is basically trivial.
In every non-empty holding set there is at least one periodic orbit, as we are able to "come back" to any starting point in positive time. Except in degenerate cases (when the holding set is a single isolated periodic orbit) there are, indeed, infinitely many distinct periodic orbits in any holding set. On the other hand, any periodic orbit is necessary (part of) a holding set. Hence the holding sets determine the periodic orbits and conversely.
It is often the case that a dynamical system has an isolated periodic trajectory. Converting this dynamical system into a control system, the single periodic trajectory may then "expand" into a "tube" (or "ring") of periodic trajectories, since we may be able to produce small perturbations of the original trajectory and still come back to the starting point. It is interesting to observe that such a tube of periodic orbits can be a holding set without any rest points.

Example1.4.3 (*Stable cycle*)
Consider in polar coordinates (ρ, θ),

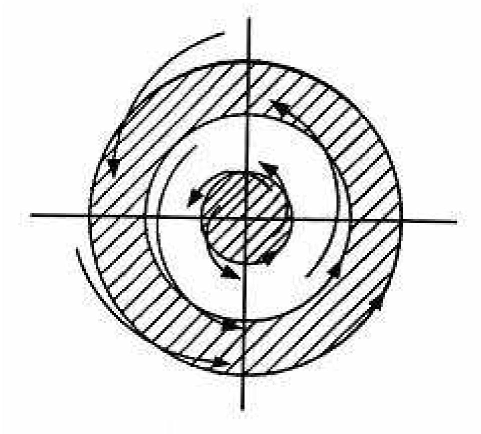
$$\dot{\rho} = \rho - \rho^2\,, \qquad \dot{\theta} = 1\,.$$

This dynamical system has an unstable critical point at the origin and the stable orbit $\rho = 1$. Converting into a control system



$$\dot{\rho} = \rho - \rho^2 + u, \quad \dot{\theta} = 1, \qquad |u(t)| \le \frac{1}{5},$$

we find that the unstable origin becomes an unstable holding set (of radius approximately 1/5), while the periodic orbit widens to a ring (of approximate width 2/5).



Chapter 2

Global optimal methods

2.1 The Hamilton-Jacobi theory

2.1.1 Introduction

In this chapter will be considered controlled systems which are continues time, finite dimensional dynamical systems with initial state x_0 in initial time t_0.
Let the model of the controlled plant is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{2.1.1}$$

where $x = (x_1, ..., x_n)^T \in R^n$ is the state vector of the system; $u = (u_1, ..., u_m) \in U \subseteq \Omega^m$ is the control vector, U is a given set of admissible piecewise continues input functions to the system; $t \in T$ is the time , $T = [t_0, t_f]$ is the time interval of the system's functioning; f is continuously differentiable vector-function,

$$f(t, x, u) = (f_1(t, x, u), ..., f_n(t, x, u))^T$$

and

$$f(t, x, u) : T \times R^n \times U \to R^n .$$

Here R^n is n-dimensional Euclidian space. The boundary condition for the equation (2.1.1) is

$$x(t_0) = x_0 \tag{2.1.2}$$

and expresses that the initial state is given at a known initial time. The final time is determined as the moment when the system reaches a given set $\Gamma \subseteq \{(t, x) \mid t > t_0, x \in R^n\}$, of admissible final events, i.e.

$$(t_f, x(t_f)) \in \Gamma \tag{2.1.3}$$

Let us define a set of admissible processes $D(t_0, x_0)$ as a set of triads $d = (t_f, x(\cdot), u(\cdot))$, where $x(\cdot) \in R^n$ is continuous and piecewise differentiable function and $u(\cdot) \in U$, satisfying eq. (2.1.1)-(2.1.3). On this set we define a performance index $I(d)$ to be minimized as the sum of an integral type term with a term which is a function of the final event

$$I(d) = \int_{t_0}^{t_f} f^0(t, x(t), u(t))dt + F(t_f, x(t_f)) \tag{2.1.4}$$

where $f^0(t,x,u)$ and $F(t_f,x)$ are given continuously differentiable functions. The performance index depends on the initial state, on the initial and final times and on the whole time history of the state and the control variables, i.e. $I(t_0,t_f,x_0,x(\cdot),u(\cdot))$.

Problem 2.1.1 (Optimal control problem)
Determine a triad $d^*=(t_f^*,x^*(\cdot),u^*(\cdot))$ such that

$$I(d^*)=\min_{d\in D(t_0,x_0)} I(d). \qquad (2.1.5)$$

2.1.2 Global sufficient conditions

Let denote with $\varphi(t;t_0,x_0,u(\cdot))$ the solution at time *t* of equations (2.1.1)-(2.1.2).

Definition 2.1.1 An *admissible control* relative to (t_0,x_0) for the system (2.1.1) and the set Γ is the control $u(\cdot)\in U\subseteq\Omega^m$ defined on the interval $[t_0,t_f]$, $t_f\geq t_0$ such that

$$(t_f,\varphi(t_f;t_0,x_0,u(\cdot)))\in\Gamma.$$

Definition 2.1.2 An *optimal control* $u^0(\cdot)$ relative to (t_0,x_0) for the system (2.1.1), performance index (2.1.4) and the set Γ is an admissible control on the interval $[t_0,t_f^0]$, $t_f^0\geq t_0$ such that

$$I(t_f^0,\varphi(\cdot;t_0,x_0,u^0(\cdot)),u^0(\cdot))\leq I(t_f,\varphi(\cdot;t_0,x_0,u(\cdot)),u(\cdot)).$$

Within the frame of optimal control theory a function built up from the system to be controlled and the integral part of the performance criterion plays a fundamental role. This is the Hamiltonian function.

Definition 2.1.3 The *Hamiltonian function* relative to the system (2.1.1) and the performance index (2.1.4) is the function

$$H(t,x,u,\lambda)=f^0(t,x,u)+\lambda' f(t,x,u) \qquad (2.1.5)$$

where $\lambda\in R^n$.

Definition 2.1.4 The Hamiltonian function is said to be *regular* if, as a function of *u*, it admits for each $t\geq t_0$, *x*, λ a unique absolute minimum $u_H^0(t,x,\lambda)$, i.e.

$$H(t,x,u_H^0(t,x,\lambda),\lambda)<H(t,x,u,\lambda),\quad \forall u\neq u_H^0(t,x,\lambda),\ \forall x\in R^n,\ \forall t\geq t_{0,}\ \ \forall\lambda\in R^n \qquad (2.1.6)$$

Definition 2.1.5 Let the Hamiltonian function be regular. The function u_H^0, which verifies the equation (2.1.6) is said to be the *H-minimizing control*.

For regular Hamiltonian function, the partial differential equation

$$\frac{\partial V(t,z)}{\partial t} + H(t,z,u_H^0(t,z,(\frac{\partial V(t,z)}{\partial z})',(\frac{\partial V(t,z)}{\partial z})') = 0 \quad (2.1.7)$$

is the Hamiltonian-Jacobi equation (HJE).
The following theorem gives a sufficient condition of optimality.

Theorem 2.1.1 Let the Hamiltonian function (2.1.5) be regular and u^0 defined on the interval $[t_0, t_f^0]$, $t_f^0 \geq t_0$, be an admissible control relative to the (t_0, x_0), so that $(t_f^0, x^0(t_f^0)) \in \Gamma$, where $x^0(\cdot) = \varphi(\cdot; t_0, x_0, u^0(\cdot))$. Let *V* be a solution of eq. (2.1.7) such that:

(i) it is continuously differentiable;
(ii) $V(t,z) = F(t,z), \ \forall (t,z) \in \Gamma$;
(iii) $u^0(t) = u_H^0(t, x^0(t), (\frac{\partial V(t,z)}{\partial z})'\Big|_{z=x^0(t)}), \ t_0 \leq t \leq t_t^0$.

Then it follows that

(j) $u^0(\cdot)$ is an optimal control relative to (t_0, x_0);
(jj) $I(t_0, x_0, x^0(\cdot), t_f^0, u^0(\cdot)) = V_0(t_0, x_0)$.

Theorem 2.1.1 supplies only sufficient optimality condition. For instance, if point (iii) fails to hold corresponding to a certain solution of the HJE, we cannot claim that the control at hand is not optimal. This theorem provides a mean of checking only whether a given control is optimal: however it can be restated to allow us to determine an optimal control. This is shown in the following lemma.

Corollary 2.1.1 Let the Hamilton function (2.1.5) be regular and *V* be a solution of the HJE (2.1.7) such that:

(i) it is continuously differentiable;
(ii) $V(t,z) = F(t,z), \quad \forall (t,z) \in \Gamma$.

If the equation

$$\dot{x}(t) = f(t, x(t), u_h^0(t, x(t), [\frac{\partial V(t,z)}{\partial z}]'\Big|_{z=x(t)})), \qquad x(t_0) = x_0$$

admits a solution x_c such that, for some $\tau \geq t_0$ $(\tau, x(\tau)) \in \Gamma$, then

$$u^0(t) = u_h^0(t, x_c(t), [\frac{\partial V(t,z)}{\partial z}]'\Big|_{z-x_c(t)})$$

is an optimal control relative to (t_0, x_0).

2.2 The LQR problem

2.2.1 Introduction

The linear quadratic regulator (LQR) problem is the most celebrated optimal control problem. It refers to a linear system and a quadratic performance index according to the following statement

Problem 2.2.1 For the system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0\end{aligned}, \tag{2.2.1}$$

where t_0 and x_0 are given, find a control which minimizes the performance index

$$J = \frac{1}{2}\{\int_{t_0}^{t_f}[x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + x'(t)Sx(t)\}. \tag{2.2.2}$$

The final time t_f is given, while no constraints are imposed on the final state $x(t_f)$. In equations. (2.2.1), (2.2.2) $A(\cdot)$, $B(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are continuously differentiable functions and $Q(t) = Q'(t) \geq 0$, $R(t) = R'(t) > 0$, for $\forall t \in [t_0, t_f]$, $S = S' \geq 0$.

LQR problem can arise in a fairly spontaneous way. Consider for example a dynamic system Σ and denote by $x_n(\cdot)$ its nominal state response one wishes to obtain. Let $u_n(\cdot)$ be the corresponding input when the system exhibits these nominal conditions $\Sigma = \Sigma_n$. Uncertainties in the system description and disturbances acting on the system lead to a closed loop configuration. The controller Σ_R, we have to design contains a system K with input the deviation δx of the actual state from x_n that supplies the correction δu to u_n in order to make δx small. There is no requirement for large corrections δu and the objective for the desired controller can be stated in terms of looking for the minimization of a quadratic performance index like the one given in eq. (2.2.2). Therefore, if the deviations δx and δu are small and Σ is described to be $\dot{x} = f(t, x, u)$ with f sufficiently regular, the effect of δx and δu can be evaluated through the linear equation

$$\delta\dot{x} = \frac{\partial f(t, x, u_n(t))}{\partial x}\Big|_{x - x_n(t)} \delta x + \frac{\partial f(t, x_n(t), u)}{\partial u}\Big|_{u = u_n(t)} \delta u .$$

Problem 2.2.1 is a particular case of Problem 2.1.1 and can be approached via the Hamilton-Jacobi theory. According to circumstances, LQR problem can be stated on a finite or infinite time interval.

2.2.2 Finite control horizon

The following result holds for the LQR problem over a finite horizon

Theorem 2.2.1 The problem 2.2.1 admits a solution for any initial state x_0 and for any finite control interval $[t_0, t_f]$. The solution is given by the control law

$$u_c^0(t,x) = -R^{-1}(t)B'(t)P(t)x\,, \tag{2.2.3}$$

where the matrix P solves the differential Riccati equation (DRE)

$$\dot{P}(t) = -P(t)A(t) - A'(t)P(t) + P(t)B(t)R^{-1}(t)B'(t)P(t) - Q(t) \tag{2.2.4}$$

with boundary condition

$$P(t_f) = S\,. \tag{2.2.5}$$

The minimal value of the performance index is

$$j^0(t_0,x_0) = \frac{1}{2}x_0'P(t_0)x_0\,.$$

Theorem 2.2.1 gives the solution to Problem 2.2.1 in terms of optimal control law. If the optimal control u^0 is necessary for a given initial state, it is sufficed to find a solution x^0 of the equation

$$\dot{x} = (A - BR^{-1}B'P)x \qquad \text{with} \qquad x(t_0) = x_0\,,$$

that computes the response of the optimal closed loop system (2.2.1),(2.2.3) and the optimal control

$$u^0(t) = -R^{-1}(t)B'(t)P(t)x^0(t)\,.$$

Remark 2.2.1 The value of the ***coefficient*** in front of the performance index (2.2.1) is not important. It can be seen as a scale factor only.

Remark 2.2.2 Theorem 2.2.1 states that the solution of the LQR problem is ***unique***.

Remark 2.2.3 A more general statement of the LQR problem can be obtained by adding ***linear*** functions of the control and/or state variables into the performance index, which thus becomes

$$J = \frac{1}{2}\int_{t_0}^{t_f}[x'(t)Q(t)x(t) + u'(t)R(t)u(t) + 2h'(t)x(t) + 2k'(t)u(t)]dt + \frac{1}{2}[x'(t_f)Sx(t_f) + 2m'x(t_f)]\,,$$

where Q, R, S are as in Problem 2.2.1 and h, k are vectors of continuously differentiable functions. The resulting optimal control admits a solution for all initial states x_0 and for all finite control intervals $[t_0,t_f]$. The solution is given by the control law

$$u_c^0(x,t) = -R^{-1}(t)\{B'(t)[P(t)x + w(t)] + k(t)\}\,,$$

where P is the solution of (2.2.4), (2.2.5), while w is the solution of the linear differential equation

$$\dot{w} = -(A = BR^{-1}B'P)'w + PBR^{-1}k - h \qquad \text{with } w(t_f) = m\,.$$

The optimal value of the performance index is

$$J^0(x_0,t_0) = \frac{1}{2}x_0'P(t_0)x_0 + w'(t_0)x_0 + v(t_0)\,,$$

where v is the solution of the linear differential equation

$$\dot{v} = \frac{1}{2}(B'w + k)'R^{-1}(B'w + k) \qquad \text{with} \qquad v(t_f) = 0 .$$

Remark 2.2.4 A different ***extension*** of the LQR problem consists in adding the term $2x'(t)Z(t)u(t)$ to the integral part of the performance index, where Z is a continuously differentiable function. Notice that the presence of this new term may substantially modify the nature of the problem, as the assumptions in Problem 2.2.1 are no longer sufficient to guarantee the existence of a solution. In fact, let

$$v := u + R^{-1}Z'x$$

so that the system description and the performance index become

$$\dot{x} = (A - BR^{-1}Z')x + Bv := A_c x + Bv$$

$$x'Qx + 2x'Zu + u'Ru = x'(Q - ZR^{-1}Z')x + v'Rv := x'Q_c x + v'Rv .$$

Thus, the original problem is transformed into a customary problem where, however, the matrix Q_c might no longer be positive semidefinite. Hence, the existence of the solution of the DRE for arbitrary finite intervals is not ensured unless a further assumption of the kind

$$\begin{bmatrix} Q & Z \\ Z' & R \end{bmatrix} \geq 0, \qquad R > 0$$

is added. Anyway, if the DRE

$$\dot{P} = -PA_c - A_c'P + PBR^{-1}B'P - Q_c \qquad \text{with} \qquad P(t_f) = S$$

admits a solution over the interval $[t_0, t_f]$ also when Q_c is not positive semidefinite, then the control law

$$u_c^0(x,t) = -R^{-1}(t)[B'(t)P(t) + Z'(t)]x$$

is optimal.

Remark 2.2.5 The assumptions on the ***sign*** of Q and S are conservative. When these assumptions are not met with, there are cases where the DRE still admits a solution and cases where the solution fails to exists over the whole given finite interval.

Remark 2.2.6 (*Tracking problem*) A third extension of the LQR problem calls for adding to the controlled system an output

$$y(t) = C(t)x(t)$$

and considering the performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} \{[y'(t) - \mu'(t)]\hat{Q}(t)[y(t) - \mu(t)] + u'(t)R(t)u(t)\}dt + \frac{1}{2}[y'(t_f) - \mu'(t_f)]\hat{S}[y(t_f) - \mu(t_f)].$$

Here C, $\hat{Q} = \hat{Q}' > 0$ and $R = R' > 0$ are continuously differentiable functions, $\hat{S} = \hat{S}' > 0$ and μ is a vector of given continuous functions. The aim is thus to make some linear combinations of the state variables behave in the way specified by μ. This optimal control problem admits a solution for each finite interval $[t_0, t_f]$, initial state $x_0(t)$ and $\mu(\cdot)$. The solution is given by the control law

$$u_c^0(x,t) = -R^{-1}(t)B'(t)[P(t)x + w(t)], \tag{2.2.6}$$

where P is the solution of the DRE

$$\dot{P} = -PA - A'P + PBR^{-1}B'P - C'\hat{Q}C \qquad \text{with} \quad P(t_f) = C'(t_f)\hat{S}C(t_f), \tag{2.2.7}$$

while w is the solution of the differential equation

$$\dot{w} = -(A - BR^{-1}B'P)'w + C'\hat{Q}\mu \qquad \text{with} \quad w(t_f) = -C'(t_f)\hat{S}\mu(t_f). \tag{2.2.8}$$

The optimal value of the performance index is

$$J^0(x_0, t_0) = \frac{1}{2}x_0{}'P(t_0)x_0 + w'(t_0)x_0 + v(t_0), \tag{2.2.9}$$

where v is the solution of the differential equation

$$\dot{v} = \frac{1}{2}(w'BR^{-1}B'w - \mu'\hat{Q}\mu) \qquad \text{with} \quad v(t_f) = \frac{1}{2}\mu'(t_f)\hat{S}\mu(t_f) \tag{2.2.10}$$

Remark 2.2.7 The tracking problem can be set also for systems which are ***not strictly proper***, that is systems where the output variable is given

$$y(t) = C(t)x(t) + D(t)u(t)$$

where C and D are matrices of continuously differentiable functions. The performance index is

$$J = \frac{1}{2}\int_{t_0}^{t_f} \{[t'(t) - \mu'(t)]\hat{Q}(t)[y(t) - \mu(t)] + u'(t)\hat{R}(t)u(t)\}dt,$$

where both matrices $\hat{Q}$ and $\hat{R}$ are symmetric, positive definite, continuously differentiable and μ is a given continuous function. The adopted performance index is purely integral. This choice simplifies the subsequent discussion without substantially altering the nature of the problem. The solution is given in terms of the control law

$$u_c^0(x,t) = -R^{-1}(t)\{[D'(t)\hat{Q}(t)C(t) + B'(t)P(t)]x + B'(t)w(t)k(t)\}$$

where $R := D'\hat{Q}D + \hat{R}$, $k := -D'\hat{Q}\mu$, P solves the DRE

$$\dot{P} = -PA_c - A_c{}'P + PBR^{-1}B'P - Q \qquad \text{with} \qquad P(t_f) = 0$$

and w is the solution of the equation

$$\dot{w} = -(A_c - BR^{-1}B'P)'w + PBR^{-1}k - h \qquad \text{with} \qquad w(t_f) = 0.$$

In these two differential equations

$$Q := C'(\hat{Q} - \hat{Q}DR^{-1}D'\hat{Q})C, \quad A_c := A - BR^{-1}D'\hat{Q}C, \quad h := C'\hat{Q}(DR^{-1}D'\hat{Q} - I)\mu.$$

The optimal value of the performance index is

$$J^0(x_0) = \frac{1}{2}x_0{}'P(t_0)x_0 + w'(t_0)x_0 + v(t_0) + \frac{1}{2}\int_{t_0}^{t_f}\mu'(t)\hat{Q}\mu(t)dt$$

Where v solves the equation

$$\dot{v} = \frac{1}{2}(B'w + k)'R^{-1}(B'w + k) \qquad \text{with} \qquad v(t_f) = 0 .$$

Remark 2.2.8 Frequently it is also convenient to prevent the first derivative of the control variable from taking on high values. This requirements can be cast into the problem formulation by adding to the integral part of the performance index the term $\dot{u}(t)\hat{R}(t)\dot{u}(t)$. If the matrix $\hat{R}$ is positive definite and continuously differentiable, the problem can be brought back to a standard LQR problem by viewing *u* as a further state variable satisfying the equation

$$\dot{u}(t) = v(t)$$

and letting *v* be a new control variable. Thus, the given problem is equivalent to the LQR problem defined on the system

$$\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{B}v(t)$$

and the performance criterion

$$J = \int [\hat{x}'(t)\hat{Q}(t)\hat{x}(t) + v'(t)\hat{R}v(t)]dt + \hat{x}'(t_f)\hat{S}\hat{x}(t_f) ,$$

where $\hat{x}(t) := [x'(t) \quad u'(t)]'$, $\hat{A}(t) := \begin{bmatrix} A(t) & B(t) \\ 0 & 0 \end{bmatrix}$,. $\hat{B} := \begin{bmatrix} 0 \\ I \end{bmatrix}$,. $\hat{Q}(t) := \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix}$, $\hat{S} := \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$.

The solution is given by the control law

$$v_c^0 = -\hat{R}^{-1}(t)\hat{B}'\hat{P}(t)\hat{x} := K_x(t)x + K_u(t)u ,$$

$\hat{P}$ being the solution of the DRE

$$\dot{P} = -P\hat{A} - \hat{A}'P + P\hat{B}\hat{R}^{-1}\hat{B}'P - \hat{Q} \qquad \text{with} \qquad P(t_f) = \hat{S} .$$

The resulting controller is no longer a purely algebraic system as in the standard LQR context but rather a dynamic system the order of which equals the number of the control variables.

Remark 2.2.9 The LQR problem can be stated also in a ***stochastic*** framework by allowing both the initial state and the input to the system to be uncertain. More precisely, assume that the controlled system is described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + v(t) \\ x(t_0) &= x_0 \end{aligned} ,$$

where *v* is a zero mean Gaussian white noise with intensity *V* and x_0 is a Gaussian random variable with expected value $\bar{x}_0$ and variance matrix Π_0 . Furthermore, it is assumed that x_0 is independent from *v*. The performance index to be minimized is

$$J_s = E\left[\int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + x'(t_f)Sx(t_f)\right],$$

where $Q \geq 0$, $S \geq 0$ and $R > 0$. If the state can be measured then the solution of the problem is constructed by the same control law which is optimal for its deterministic version (2.2.3)-(2.2.5). In fact, corresponding to the control law $u(x,t) = K(t)x$ the value taken by the index J_s is

$$J_s = tr[P_K(t_0)(\Pi_0 + \bar{x}_0\bar{x}_0') + \int_{t_0}^{t_s} VP_K(t)dt].$$

Matrix P_K is the solution of the Lyapunov differential equation

$$P - P(A+BK) - (A+BK)'P - (Q + K'RK) \text{ with } P(t_f) = S.$$

If $V = 0$ and $\Pi_0 = 0$ then $\bar{x}_0' P_K(t_0)\bar{x}_0 \geq \bar{x}_0' P(t_0)\bar{x}_0$, where *P* is the solution of (2.2.4),(2.2.5). Since $\bar{x}_0$ and t_0 are both arbitrary, it follows that $P_K(t) \geq P(t)$, $t \in [t_0, t_f]$. Obviously $P_K(\cdot) = P(\cdot)$ when $K = -R^{-1}B'P$ so that J_s is minimized by this choice of *K*. Indeed, $tr[P_K\Delta] \geq tr[P\Delta]$ for all $\Delta = \Delta' \geq 0$ (or $tr[(P_K - P)\Delta] \geq 0$) since the eigenvalues of the product of two positive semidefinite matrices are real and nonnegative.

2.2.3 Infinite control horizon

By no means can the linear-quadratic optimal control problem over an *infinite* horizon be viewed as a trivial extension of the problem over a finite horizon, which has been considered to some extent in the previous section. As a matter of fact, the assumption which have proved to be sufficient in the later case are no longer such in the former one.
Problem 2.2.1 will not be discussed for $t_{f=\infty}$ and $S = 0$. This particular choice for *S* is justified mainly by the fact that in the most significant class of LQ problems over an infinite horizon, the state asymptotically tends to zero and a nonintegral term in the performance index would be useless. The LQ problem over an infinite horizon is therefore stated in the following way.

Problem 2.2.2. (*Linear-quadratic problem over an infinite horizon*)
Given the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2.2.11}$$

where x_0 and t_0 are specified find a control which minimizes the performance index

$$J = \frac{1}{2}\int_{t_0}^{\infty} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt$$

The final state is free and $A(\cdot), B(\cdot), Q(\cdot), R(\cdot)$ are continuously differentiable functions; further $Q(t) = Q^{/}(t) > 0, \ \forall \ \ t \geq t_0$

A solution of this problem is provided by the following theorem.

Theorem 2.2.2 Let the system (2.2.11) be controllable for each $t \geq t_0$. Then Problem 2.2.2 admits a solution for each initial state x_0 which is specified by the control low

$$u_c^0(x,t) = -R^{-1}(t)B'(t)\bar{P}(t)x \tag{2.2.12}$$

where

$$\bar{P}(t) := \lim_{t_j \to \infty} P(t, t_f), \tag{2.2.13}$$

$P(\cdot, t_f)$ being the solution of the differential Riccati equation

$$\dot{P} = -PA - A'P + PBR^{-1}B'P - Q \tag{2.2.14}$$

satisfying the boundary condition

$$P(t_f, t_f) = 0, \quad t_0 < t_f < \infty. \tag{2.2.15}$$

Further, the optimal value of the performance index is

$$J^0(x_0, t_0) = \frac{1}{2} x_0' \bar{P}(t_0) x_0 \tag{2.2.16}$$

2.2.4 The optimal regulator

Due to the importance of the results and the number of applications, the LQ problem over an infinite horizon when both the system and the performance index are time-invariant, that is when A, B, Q, R are constant matrices is particularly meaningful. The resulting problem is usually referred to us the *optimal regulator problem* and apparently is a special case of the previously considered LQ problem over an infinite horizon. However, it is worth discussing it in detail since independence of data from time implies a substantial simplification of the relevant results, making their use extremely simple. Thus the problem at hand is

Problem 2.2.3. (*Optimal regulator problem*).
For the time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0 \end{aligned} \tag{2.2.17}$$

where x_0 is given, find a control that minimizes the performance index

$$J = \tfrac{1}{2}\int_0^\infty [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt \tag{2.2.18}$$

The final set is unconstrained and $Q = Q' \geq 0$, $R = R' > 0$.

Observe that, thanks to time-invariance, the initial time has been set to 0 without loss of generality. The following result holds for the problem above.

Theorem 2.2.3 Let the pair (A, B) be reachable. Then the Problem (2.2.3.) admits a solution for each x_0. The solution is specified by the control low

$$u_{cs}^{0}(x) = -R^{-1}B'\bar{P}x \tag{2.2.19}$$

where $\bar{P} = \bar{P}' \geq 0$ solves the algebraic Riccati equation (ARE)

$$PA + A'P - PBR^{-1}B'P + Q = 0 \tag{2.2.20}$$

and is such that

$$\bar{P} = \lim_{t_f \to \infty} P(t, t_f),$$

$P(\cdot, t_f)$ being the solution of the differential Riccati equation $\dot{P} = -PA - A'P + PBR^{-1}B'P - Q$ with boundary condition $P(t_f, t_f) = 0$. Further, the optimal value of the performance index is

$$J^{0}(x_0) = \frac{1}{2}x_0'\bar{P}x_0 \tag{2.2.21}$$

Remark 2.2.11 *(Control in the neighborhood of an equilibrium point)*
From a particular point of view the importance of the optimal regulator problem is considerably enhanced by the discussion at the beginning of this chapter. Indeed equation (2.2.17.) can be seen as resulting from the linearization of the controlled system about an equilibrium state, say ξ_n. For this system the state ξ is desired to be close to such a point, without requiring, however, large deviations of the control variables η from the value η_n which, in nominal conditions, produces ξ_n. In this perspective, *x* and *u* are with reference to the quoted equation, the state and control deviations, respectively, and the meaning of the performance index is obvious. Further, should the control low (2.2.19) force the state of system (2.2.17) to tend to 0 corresponding to any initial state, than it would be possible to conclude that the system has been *stabilized* in the neighborhood of the considered equilibrium.

Theorem 2.2.4 Assume that the pair (A, B) is reachable and let P_a be any element of the set P. Then $P_a - \bar{P} \geq 0$.

Lemma 2.2.1 Let *Q* be a symmetric positive semidefinite matrix and C_1 and C_2 two distinct factorizations of it. Let *A* be a square matrix with the same dimension as *Q*. Then the unobservable subspace of the pair (A, C_1) coincides with the unobservable subspace of the pair (A, C_2).

Theorem 2.2.5 Problem 2.2.3 admits a solution for each initial state x_0 if and only if the observable but unreachable part or the triple (A, B, Q) is asymptotically stable.

Remark 2.2.12 *(Decomposition of the ARE)*
If the triple (A, B, C) is not ***minimal***, the ARE to be taken into account simplifies a lot. In fact, the canonical decomposition of the triple induces a decomposition of the equation as well, thus enabling us to set some parts of the solution to zero. More precisely, assume that *A*, *B* and *C* are already in canonical form, namely

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_5 & 0 & A_6 \\ 0 & 0 & A_7 & A_8 \\ 0 & 0 & 0 & A_9 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & C_1 & 0 & C_2 \end{bmatrix}$$

and partition matrix P according to letting

$$P := \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ P_2' & P_5 & P_6 & P_7 \\ P_3' & P_6' & P_8 & P_9 \\ P_4' & P_7' & P_9' & P_{10} \end{bmatrix}.$$

From the differential equations for the P_i's it follows that $P_i(\cdot, t_f) = 0, \; i = 1,2,3,4,6,8,9$ while the remaining blocks solve the three equations

$$\dot{P}_5 = -P_5 A_5 - A_5' P_5 + P_5 B_2 R^{-1} B_2' P_5 - C_1' C_1,$$

$$\dot{P}_7 = -P_7 A_9 - (A_5' - P_5 B_2 R^{-1} B_2') P_7 - P_5 A_6 - C_1' C_2$$

$$\dot{P}_{10} = -P_{10} A_9 - A_9' P_{10} - P_7' A_6 - A_6' P_7 + P_7' B_2 R^{-1} B_2' P_7 - C_2' C_2$$

which sequentially can be managed. The only nonlinear equation is the first one, which already is the Riccati equation for Problem 223 relative to the reachable and observable part of the triple (A, B, C). The two remaining equations are linear. Thus

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{P}_5 & 0 & \overline{P}_7 \\ 0 & 0 & 0 & 0 \\ 0 & \overline{P}_7' & 0 & \overline{P}_{10} \end{bmatrix}$$

where $\overline{P}_i, \; i = 5,7,10$ are the limiting values (as $t_f \to \infty$) of the solutions of the above equations with boundary conditions $P_i(t_f, t_f) = 0$. These matrices are solutions of the algebraic equations which are obtained from the differential ones by setting the derivatives to zero and substituting for P_5 and P_7 their limiting values. The next section will show that $\overline{P}_5$ is such that $A_5 - B_2 R^{-1} B_2' \overline{P}_5$ is *stable* (all its eigenvalues have negative real parts). This fact implies that two linear algebraic equations which determine $\overline{P}_7$ and $\overline{P}_{10}$ admit a unique solution. Indeed both of them are of the form $XF + GX + H = 0$ with F and G stable. Thus the solution of Problem 2.2.3 (when it exists relative to any initial state) can be found by first computing $\overline{P}_5$, solution of the ARE (in principal, by exploiting Theorem 2.2.4 , actually by making reference to the result in Chapter 4) and subsequently determining the (unique) solutions $\overline{P}_7$ and $\overline{P}_{10}$ of the remaining two linear equations.

Finally, if the given triple (A,B,C) is not in canonical form (resulting from a change of variables defined by a nonsingular matrix *T*) the solution of the problem relies on $P_{or} := T'\overline{P}T$. The check of this claim is straightforward.

Remark 2.2.13 (*Tracking problem over an infinite horizon*) The optimal tracking problem in Remark 2.2.6 with reference to a finite control interval can be stated also for an infinite time horizon. This extension is particularly easy if the problem at hand is time-invariant (the matrices which define both the systems and the performance index are constant) and the signal to be tracked is the output of a linear time-invariant system. Under these circumstances the optimal control problem is specified by

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$
$$x(0) = x_0$$

and

$$J = \int_0^\infty \{[y'(t) - \mu'(t)]\hat{Q}[y(t) - \mu(t)] + u'(t)Ru(t)\}dt$$

where μ is an output of the dynamic system

$$\dot{\vartheta}(t) = F\vartheta(t),$$
$$\mu(t) = H\vartheta(t),$$
$$\vartheta(0) = \vartheta_0.$$

As in Remark 2.2.6, $\hat{Q} = \hat{Q}' > 0$. Further, due to self-explanatory motivations, the pair (F,G) is assumed to be observable so that if x_0 and ϑ_0 are generic though given, asymptotic stability of *F* must be required. Under these circumstances, it is not difficult to verify that the solution of the problem exists for each x_0 and ϑ_0 if and only if the observable but unreachable part of the triple (A,B,C) is asymptotically stable. The solution can be deduced by noticing that the problem at hand can be given the form of Problem 2.2.3 provided 6that the new system

$$\dot{\xi} = W\xi + Vu$$

and the performance index

$$J = \int_0^\infty (\xi'\Theta\xi + u'Ru)dt$$

are considered, where

$$\xi := \begin{bmatrix} x \\ \vartheta \end{bmatrix}, \quad W := \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}, \quad V := \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \Theta := \begin{bmatrix} C'\hat{Q}C & -C'\hat{Q}H \\ -H'\hat{Q}C & H'\hat{Q}H \end{bmatrix}.$$

Thus the optimal control low is

$$u_c^0(x,t) = -R^{-1}B'(\overline{P}_1 x + \overline{P}_2\vartheta(t))$$

where $\overline{P}_i$ solves the ARE

$$0 = PA + A'P - PBR^{-1}B'P + C'\hat{Q}C$$

and is such that

$$\overline{P}_1 = \lim_{t_f \to \infty} P(t, t_f), \; P(t, t_f)$$

being the solution of the DRE

$$\dot{P} = -PA - A'P + PBR^{-1}B'P - C'\hat{Q}C$$

satisfying the boundary condition $P_i(t_f, t_f) = 0$, while $\overline{P}_2$ solves the linear equation

$$0 = PF + (A - BR^{-1}B'\overline{P}_1)'P - C'\hat{Q}H .$$

Finally, the optimal value of the performance index is

$$J^0(x_0, \vartheta_0) = x_0'\overline{P}_1 x_0 + 2\vartheta_0'\overline{P}_2' x_0 + \vartheta_0'\overline{P}_3\vartheta_0 ,$$

where $\overline{P}_3$ is the solution of the Lyapunov equation

$$0 = PF + F'P - \overline{P}_2'BR^{-1}B'\overline{P}_2 + H'\hat{Q}H .$$

Remark 2.2.14 (*Penalties of the control derivatives*) The discussion in Remark 2.2.8 is steel valid in the case $t_f = \infty$ even if some care must be paid to existence of the solution. With the notation adopted, let Λ_{nr} and $\hat{\Lambda}_{nr}$ be the spectra of the unreachable parts of the pairs (A, B) and $(\hat{A}, \hat{B})$, respectively. Then $\Lambda_{nr} = \hat{\Lambda}_{nr}$. In fact, if T_r is a nonsingular matrix which performs the canonical decomposition of the pair (A, B) into the reachable and unreachable parts, namely a matrix such that

$$T_r A T_r^{-1} = \begin{bmatrix} A_{1r} & A_{2r} \\ 0 & A_{3r} \end{bmatrix}, \quad T_r B = \begin{bmatrix} B_{1r} \\ 0 \end{bmatrix}, \quad (A_{1r}, B_{1r}) = \text{reachable},$$

then Λ_{nr} is the spectrum of A_{3r}. By letting

$$\hat{T}_r := \begin{bmatrix} T_r & 0 \\ 0 & I \end{bmatrix}$$

we obtain

$$\hat{T}_r A \hat{T}_r^{-1} = \begin{bmatrix} A_{1r} & A_{2r} & B_{1r} \\ 0 & A_{3r} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{T}_r \hat{B} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

so that $\Lambda_{nr} \subseteq \hat{\Lambda}_{nr}$, since the spectrum of A_{3r} is a subset of $\hat{\Lambda}_{nr}$. It is not difficult to verify that

$$\left(\begin{bmatrix} A_{1r} & B_{1r} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix} \right) = \text{reachable}$$

from which $\hat{\Lambda}_{nr} = \Lambda_{nr}$. Indeed, if such a pair is not reachable, it follows that,

$$\begin{bmatrix} A_{1r}' & 0 \\ B_{1r}' & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \lambda \begin{bmatrix} x \\ u \end{bmatrix}, \quad \begin{bmatrix} x \\ u \end{bmatrix} \neq 0, \quad \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

These equations imply that $u = 0$ and, the pair (A_{1r}, B_{1r}) should not be reachable.

Let now Λ_{no} and $\hat{\Lambda}_{no}$ be the spectra of the unobservable parts of the pairs (A,C) and $(\hat{A},\hat{C})$, respectively, where $\hat{C}'\hat{C} = \hat{Q} := diag[C'C, D'D]$, C and D being factorizations of Q and R, respectively. Then $\Lambda_{no} \subseteq \hat{\Lambda}_{no}$. In fact, let T_o be a nonsingular matrix which performs the canonical decomposition of the pair (A,C) into the observable and unobservable parts, namely a matrix such that

$$T_o A T_o^{-1} = \begin{bmatrix} A_{1o} & 0 \\ A_{2o} & A_{3o} \end{bmatrix}, \quad C T_o^{-1} = \begin{bmatrix} C_{1o} & 0 \end{bmatrix}, \quad (A_{1o}, C_{1o}) = \text{observable}.$$

Then, Λ_{no} is the spectrum of A_{3o}. If

$$\hat{T}_o := \begin{bmatrix} T_o & 0 \\ 0 & I \end{bmatrix}$$

we obtain

$$\hat{T}_o A \hat{T}_o^{-1} = \begin{bmatrix} A_{1o} & 0 & B_{1o} \\ A_{2o} & A_{3o} & B_{2o} \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{C}\hat{T}_o^{-1} = \begin{bmatrix} C_{1o} & 0 & 0 \\ 0 & 0 & D \end{bmatrix}$$

so that $\Lambda_{no} \subseteq \hat{\Lambda}_{no}$ since the spectrum of A_{3o} is a subset of $\hat{\Lambda}_{no}$.

Finally, denote by Λ_{nro} and $\hat{\Lambda}_{nro}$ the spectra of the unreachable but observable parts of the triples (A,B,C) and $(\hat{A},\hat{B},\hat{C})$. From the preceding discussion it can be concluded that $\hat{\Lambda}_{nro} \subseteq \Lambda_{nro}$.

If a solution of Problem 2.2.3 defined by the quadruple (A,B,Q,R) exists for each initial state $x(0)$, i.e. all elements of Λ_{nro} lie in the open left half-plane, then a solution of Problem 2.2.3, defined by the quadruple $(\hat{A},\hat{B},\hat{Q},\hat{R})$ (recall that $\hat{R}$ is the weighting matrix for $\dot{u}$ in the performance index), exists for each initial state $\begin{bmatrix} x'(0) & u'(0) \end{bmatrix}'$, since, necessarily, all elements of $\hat{\Lambda}_{nro}$ lie in the open left half-plane.

In the special case where $rank(B)$ is maximum and equal to the number of columns, the optimal regulator can be given a form different from the one which, referring to a finite control interval can anyhow be adopted also in the present context, the only significant difference being the time-invariance of the system. Since $B'B$ is nonsingular, from the system equation $\dot{x} = Ax + Bu$ it follows that

$$u = (B'B)^{-1} B'(\dot{x} - Ax).$$

On the other hand, the solution of Problem 2.2.3 implies that $\dot{u} = K_x x + K_u u$ so that

$$\dot{u} = \hat{K}_{\dot{x}}\dot{x} + \hat{K}_x x$$

where $\hat{K}_{\dot{x}} := K_u (B'B)^{-1} B'$ and $\hat{K}_x := K_x - K_u (B'B)^{-1} B' A$.

By performing the integration of the equation for $\dot{u}$ between the initial time 0 and a generic instant *t* we obtain

$$u(t) = \hat{K}_{\dot{x}}(x(t) - x(0)) + \int_0^t \hat{K}x(\tau)d\tau + u(0) .$$

This is the control low which can be interpreted as a generalization of the PI controller to the multiplicative case.

Remark 2.2.15 *(Performance evaluation of the frequency-domain)*
The synthesis procedure based on the solution of Problem 2.2.3 can easily be exploited to account for requirements (more naturally) expressed in the frequency domain, as, for instance, those calling for a weak dependence of some variables of interest on others in a specified frequency range. In other words, the presence of harmonic components of some given frequencies in some state and/or control variables must be avoided, or, equivalently, suitable penalties on them must be set. This can be done in a fairly easy way. Indeed, recall that thanks to Parceval's theorem

$$\int_0^\infty z'(t)z(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty Z^{\sim}(j\omega)Z(j\omega)d\omega$$

where *z* is a time function, *Z* is its Fourier transform and it has obviously been assumed that the written expression make sense. Therefore, a penalty on some harmonic components in the signal $x(t)$ can be set by looking for the minimization of a performance index of the form

$$\frac{1}{2\pi}\int_{-\infty}^\infty X^{\sim}(j\omega)F_x^{\sim}(j\omega)F_x(j\omega)X(j\omega)d\omega$$

where F_x is a suitable matrix of chapping function. F_x is rational and proper (not necessarily strictly proper) it can be interpreted as the transfer function of a system whit input *x*, $Z(j\omega) = F_x(j\omega)X(j\omega)$ is the Fourier transform of the output $z(t)$ and the integral of $z'z$ is the quantity to be evaluated. The usual performance index takes on the following (more general) form

$$J_f = \frac{1}{2\pi}\int_{-\infty}^\infty [X^{\sim}(j\omega)F_x^{\sim}(j\omega)F_x(j\omega)X(j\omega) + U^{\sim}(j\omega)F_u^{\sim}(j\omega)F_u(j\omega)U(j\omega)]d\omega$$

where F_x and F_u are proper rational matrices. This index has to be minimized subject to equation (2.2.17). The resulting optimal control problem can be tackled by first introducing two (minimal) realizations of F_x and F_u. Let the quadruples (A_x, B_x, C_x, D_x) and (A_u, B_u, C_u, D_u) define such realizations, respectively, and note that

$$J_f = \int_0^\infty [x'_A(t)Q_A x_A(t) + 2x'_A(t)Z_A u(t) + u'(t)R_A u(t)]dt$$

if $x_A := [x' \quad z'_x \quad z'_u]'$ whit $\dot{x}_A = A_A x_A + B_A u$ where

$$A_A := \begin{bmatrix} A & 0 & 0 \\ B_x & A_x & 0 \\ 0 & 0 & A_u \end{bmatrix}, \quad B_A := \begin{bmatrix} B \\ 0 \\ B_u \end{bmatrix}$$

and

$$Q_A := \begin{bmatrix} D_x'D_x & D_x'C_x & 0 \\ C_x'D_x & C_x'C_x & 0 \\ 0 & 0 & C_u'C_u \end{bmatrix}, \quad Z_A := \begin{bmatrix} 0 \\ 0 \\ C_u'D_u \end{bmatrix}, \quad R_A := D_u'D_u .$$

Thus the problem whit frequency domain requirements has been restated as an LQ problem over an infinite horizon where a rectangular term is present in the performance index. The results for the optimal regulator problem can be exploited provided only that the quadratic form in *u* is possible definite, namely if $\text{rank}(D_u)$ equals the number of its columns. Indeed, the state weighting matrix is positive semidefinite since the form of J_f implies that $Q_{Ac} := Q_A - Z_A R_A^{-1} Z_A' \geq 0$ (see Remark 2.2.4). Assuming that $R_A > 0$, the solution of the problem exists for each initial state $x_A(0)$ if and only if (see Theorem 2.2.5) the observable but unreachable part of the triple (A_A, B_A, C_A) is asymptotically stable, where

$$C_A := \begin{bmatrix} D_x & C_x & 0 \\ 0 & 0 & C_u \end{bmatrix}$$

is a factorization of Q_A. In conclusion, the problem whit frequency requirements can be viewed as a customary optimal regulator problem if F_x and F_u are such as to verify the above assumptions on sign and stability. Note that the resulting regulator is no longer purely algebraic. Indeed, the control variable *u* depends, through a constant matrix, upon the whole enlarged state vector x_A, so that $u = K_x x + K_{zx} z_x + K_{zu} z_u$. Thus the regulator is a dynamic system whit state $[z_x' \quad z_u']'$, input *x* (the state of the controlled system) and output *u*.

Remark 2.2.16 *(Stochastic control problem over an infinite horizon)*
The discussion in Remark 2.2.9 can be suitably modified to cover the case of an unbounded control interval. Corresponding to the (time-invariant) system

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t),$$
$$x(0) = x_0$$

where *v* and x_0 are as in Remark 2.2.9, reference can be made to either the performance index

$$J_{s1} = E\left[\int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt\right]$$

when $v(\cdot) = 0$, or the performance index

$$J_{s2} = E\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt\right]$$

when $x_0 = 0$. In both cases the solution, if it exists, is constituted by the control law (2.2.19) defined in Theorem 2.2.3. In the simple cases where $\Pi_0 + \bar{x}_0\bar{x}_0' > 0$ (performance index J_{s1}) and $V > 0$ (performance index J_{s2}), the solution exists is and only if the unreachable but observable part of the triple (A, B, C) is asymptotically stable, where *C* is such that $C'C = Q$.

The remaining part of this section is dedicated to some particular but very important properties of the optimal control feedback system and discussing the potentialities of design methods based on the minimization of quadratic indices.

2.2.4.1 Stability properties

The stability properties of system (2.2.17),(2.2.19), that is of system

$$\dot{x}(t) = (A - BR^{-1}B'\overline{P})x(t), \qquad (2.2.22)$$

are now analyzed in detail. The fact that the control law guarantees a finite value of the performance index corresponding to any initial state suggests that system (2.2.22) should be asymptotically stable if every nonzero motion of the state is *detected* by the performance index.

Lemma 2.2.2 Let the pare (A,B) be reachable and $Q = C'C$. Then the matrix $\overline{P}$ which satisfies the optimal control law for Problem 2.2.3 is positive definite if and only is the pair (A,C) is observable.

Theorem 2.2.6 Let $Q = C'C$ and the triple (A,B,C) be minimal. Then the closed loop system resulting from the solution of Problem 2.2.3 is asymptotically stable.

Theorem 2.2.7 Assume that a solution of Problem 2.2.3 exists for each initial state. Then the optimal closed loop system is asymptotically stable if and only is the pair (A,Q) is detectable.

Remark 2.2.17 (*Existence and stabilizing properties of the optimal regulator*) A summary of the discussion above concerning the existence and the stabilizing properties of the solution of Problem 2.2.3 is presented in Figure 2.2.26 where reference is made to a canonical decomposition of the triple (A,B,C) and the notation of Remark 2.2.12 is adopted. Further, the term "stab." denotes asymptotic stability and the existence or inexistence of the solution has to be meant for an arbitrary initial state.

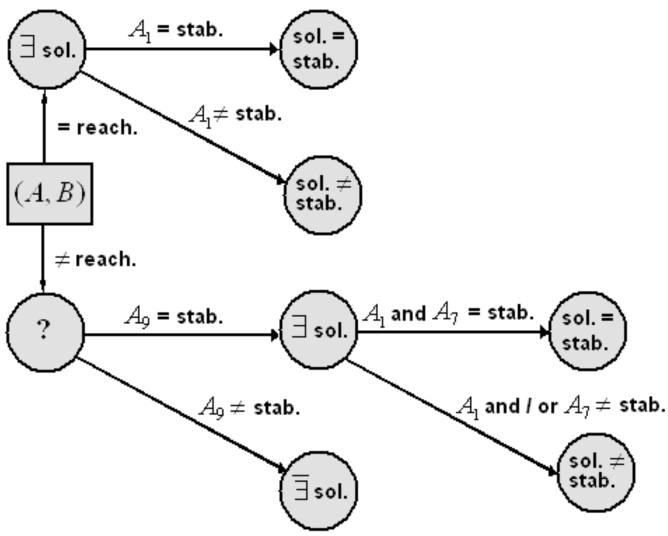


Figure 2.2.26: Optimal regulator problem: existence and stabilizing properties of the solution.

Remark 2.2.18 (*Optimal regulation with constant exogenous inputs*) The results concerning the optimal regulator can be exploited when the system has to be controlled so as to achieve asymptotic zero error regulation in the presence of unknown inputs of polynomial type. In the case of constant signals, the control system is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Md, \\ y(t) &= Cx(t) + Nd, \\ x(0) &= x_0, \end{aligned} \tag{2.2.23}$$

and a regulator has to be designed so as to guarantee for each constant signal y_s and d and each initial state x_0,

$$\lim_{t\to\infty} t(t) = y_s .$$

Within this framework y_s is the set point for y, while d accounts for the disturbances acting on the system input and output. In the present setting the triple (A, B, C) is minimal, the number of control variables equals the number of output variables and the state of the system is available to the controller. The controller can be considered as constructed by two subsystems: the first one is described by the equation

$$\dot{\xi}(t) = y_s - y(t) \tag{2.2.24}$$

while the second one has to generate the control variable u on the basis of x and ξ in such a way as to asymptotically stabilize the whole system. In designing this second system it is meaningful to ask for small deviations of the state and control variables from their steady state values together with a fast zeroing of the error. Since the first variations of the involved variables for constant inputs are described by system Σ_v obtained from equation (2.2.23),(2.2.24) by setting $d = 0$ and $y_s = 0$, namely

$$\begin{aligned} \delta\dot{x}(t) &= A\delta x(t) + B\delta u(t), \\ \delta y(t) &+ Cx(t), \\ \delta\dot{\xi}(t) &= -\delta y(t), \end{aligned}$$

a satisfactory answer is to let the second subsystem be constituted by the solution of Problem 2.2.3 for Σ_v and a suitable performance index. Thus, chosen

$$J = \int_0^\infty [\delta y'(t) Q_y \delta y(t) + \delta\xi'(t) Q_\xi \delta\xi(t) + \delta u'(t) R \delta u(t)] dt$$

with Q_ξ and R positive definite and Q_y positive semidefinite, the optimal control law, if it exists, will surely be stabilizing, since Σ_v is observable from ξ and given by

$$\delta u_{cv}^o(\delta x, \delta\xi) = K_1 \delta x + K_2 \delta\xi .$$

The existence of the solution is guaranteed by the reachability of Σ_v, namely by the fulfillment of the condition

$$n + m = rank(\begin{bmatrix} B & AB & A^2B & \cdots \\ 0 & -CB & -CAB & \cdots \end{bmatrix}) = rank(\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} 0 & B & AB & \cdots \\ I & 0 & 0 & \cdots \end{bmatrix})$$

which in turn is equivalent to saying that system $\Sigma(A,B,C,0)$ does not possess transmission zeros at the origin (actually invariant zeros, because of the minimality of Σ). In fact, reachability of the pair (A,B) implies that in the above equation the rank of the second matrix on the right-hand side be equal to $m+n$ and, in view of the already mentioned minimality of $\Sigma(A,B,C,0)$, that there are transmission zeros at the origin if and only $Ax+Bu=0$ and $Cx=0$ with x and/or u different from 0. On the other hand, if $\Sigma(A,B,C,0)$, which possesses as many inputs as outputs, has a transmission zero located at the origin, then it would follow in turn entail the existence of a zero eigenvalue in the unreachable part of Σ_v. Since this system is observable, we would conclude that no solution exists for Problem 2.2.3 when stated on such a system. Thus zero error regulation can be achieved in the presence of constant inputs only if none of the transmission zeros of $\Sigma(A,B,C,0)$ is located at the origin.

Remark 2.2.19 (*Penalties on the control derivatives*) The problem considered in Remarks 2.2.8 and 2.2.14 can be discussed further with reference to the stability properties of the solution.
First recall that in view of Theorems 2.2.5 and 2.2.7, if a solution of Problem 2.2.3 stated for the quadruple (A,B,Q,R) exists for each initial state $x(0)$ and the resulting closed loop system is asymptotically stable, then the set $\Lambda_{no}\cup\Lambda_{nro}$ must be stable, i.e. all its elements must have negative real parts. Assuming that this set is such, it is possible to claim that the Problem 2.2.3 stated for the quadruple (A,B,Q,R) admits a solution for each initial state $[x'(0) \quad u'(0)]'$ if the set $\hat{\Lambda}_{nro}$ is stable. Since $\hat{\Lambda}_{nro}\subseteq\Lambda_{nro}$, the set $\hat{\Lambda}_{nro}$ is stable if the set Λ_{nro} is such.
If stability of the resulting closed loop system has to be guaranteed as well, stability of the set $\hat{\Lambda}_{no}$ must be checked. To this end, observe that it $\lambda\in\hat{\Lambda}_{no}$, then it must be

$$\hat{A}\begin{bmatrix} x \\ u \end{bmatrix}=\lambda\begin{bmatrix} x \\ u \end{bmatrix}, \quad \hat{C}\begin{bmatrix} x \\ u \end{bmatrix}=0, \quad \begin{bmatrix} x \\ u \end{bmatrix}\neq 0 .$$

Since matrix R is positive definite, D (which is a factorization of it) is square and nonsingular without lack of generality, so that these equations imply $u=0$ since $\hat{C}=diag[C,D]$ and $\lambda\in\Lambda_{no}$. Therefore, the set $\hat{\Lambda}_{no}$ is stable if the set Λ_{no} is stable.
Notice, that the presence of a penalty term on the control derivative in the performance index allows one to relax the requirement on the sign of R: indeed it appears in matrix $\hat{Q}$ only and can be positive semidefinite. In this case, matrix D, if chosen of maximal rank, is no longer square and the above discussion about stability of the closed loop system has to be modified. In fact, assume as before that the set $\hat{\Lambda}_{nro}$ is stable in order to guarantee the existence of the solution for each initial state and, as for $\hat{\Lambda}_{no}$, note that the above equations (which amount to say $\lambda\in\hat{\Lambda}_{no}$) still imply $u=0$ if $\lambda\neq 0$: thus such an eigenvalue is currently stable if Λ_{no} is stable. If, on the other hand, $\lambda=0$, those equations become $Ax+Bu=0,\ Cx=0$. Hence a transmission zero of system $\Sigma(A,B,C,0)$ is located at the origin since x and u cannot simultaneously be zero. Therefore, if R is not positive definite, the conclusion can be drawn that Problem 2.2.3, stated for the quadruple $(\hat{A},\hat{B},\hat{Q},\hat{R})$, can not admit a stabilizing solution whenever system $\Sigma(A,B,C,0)$ possesses transmission zeros located at the origin.

Problem 2.2.4 (*Optimal regulator problem with exponential stability*) For the time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \tag{2.2.25}$$

where x_0 is given, find a control which minimizes the performance index

$$J = \int_0^\infty e^{2\alpha t}[x'(t)Qx(t) + u'(t)Ru(t)]dt\,. \tag{2.2.26}$$

No constraints are imposed on the final state and further $Q = Q' \geq 0$, $R = R' > 0$, while α is a given nonnegative real number.

For this problem the following result holds.

Theorem 2.2.8 Let the triple (A, B, Q) be minimal. Then the solution of Problem 2.2.4 exists for each initial state x_0 and each $\alpha \geq 0$. The solution is characterized by the control law

$$u^o_{cs\alpha}(x) = -R^{-1}B'\overline{P}_\alpha x \tag{2.2.27}$$

where $\overline{P}_\alpha$ is the symmetric and positive definite solution of the algebraic Riccati equation

$$0 = P(A + \alpha I) + (A + \alpha I)'P - PBR^{-1}B'P + Q \tag{2.2.28}$$

such that $\overline{P}_\alpha = \lim_{t_f \to \infty} P(t, t_f)$, where $P_\alpha(t, t_f)$ is the solution of the differential Riccati equation

$$\dot{P} = -P(A + \alpha I) - (A + \alpha I)'P + PBR^{-1}B'P - Q$$

with boundary condition

$$P_\alpha(t_f, t_f) = 0\,.$$

Further, all eigenvalues of the closed loop system (2.2.25),(2.2.28) have real parts smaller than $-\alpha$.

2.2.4.2 Robustness properties

The control law which is a solution of the optimal regulator problem has been shown to be stabilizing under suitable mind assumptions. However, this is not the only nice property that it possesses. First the following lemma is needed. Consider the system

$$\dot{x}(t) == Ax(t) + Bu(t), \tag{2.2.30}$$

$$y(t) = Cx(t)\,, \tag{2.2.31}$$

$$u(t) = -R^{-1}B'Px(t) := Kx(t) \tag{2.2.32}$$

where *P* is any symmetric solution of the ARE

$$0 = PA + A'P \_ PBR^{-1}B'P + Q \tag{2.2.33}$$

with $Q - C'C$ and $R = R' > 0$.

Lemma 2.2.3 Let *K* be given by equations (2.2.32),(2.2.33). Then

$$G^{\sim}(s)G(s) = I + H^{\sim}(s)QH(s) \qquad (2.2.34)$$

where

$$G(s) = I - R^{\frac{1}{2}}KH(s)$$

$$H(s) = (sI \_ A)^{-1}BR^{-\frac{1}{2}}.$$

From equation (2.2.34), letting $s = j\omega,\ \ \omega \in R$, it follows that $G'(-j\omega)G(j\omega) \geq I$ since its left-hand side is an hermitian matrix (actually it is the product of a complex matrix by its conjugate transpose), while its right-hand side is the sum of the identity matrix with an hermitian positive semidefinite matrix. In the particular case of a scalar control variable, equation (2.2.34) implies that

$$\left|1 - K(j\omega I - A)^{-1}B\right| \geq 1, \ \ \forall \omega \in R\,. \qquad (2.2.35)$$

Based on this relation the following theorem states that the optimal closed loop system is robust of both *phase and gain margin*.

Theorem 2.2.9 Consider the system (2.2.30),(2.2.31) and assume that:

i) The input *u* is scalar;
ii) The triple (A,B,C) is minimal;
iii) In equation (2.2.32) $P = \overline{P}$, the solution of equation (2.2.33) relevant to Problem 2.2.3 defined by the quadruple $(A,B,C'C,R)$.

Then the phase margin of the closed loop system is not less than $\pi/3$ while the gain margin is infinite.

Theorem 2.2.10 Consider Problem 2.2.3 and assume that the pair (A,B) is reachable, $Q > 0$ and $R > 0$ diagonal. Then each one of the *m* loops of the system resulting from the implementation of the optimal control law possesses a phase margin not smaller than $\pi/3$ and an infinite gain margin.

2.2.4.3 Cheap control

The behaviour of the solution of the optimal regulator problem is now analyzed when the penalty set on the control variable becomes less important, that is when it is less mandatory to keep the input values at low levels. When the limit situation is reached where no cost is associated to the use of the control variable (the control has become a cheap item) the input can take on arbitrary high values and the ultimate capability of the system to follow the desired behaviour (as expressed by the performance index) is put into evidence.
Consider the system

$$\dot{x}(t) == Ax(t) + Bu(t), \qquad (2.2.36a)$$

$$y(t) = Cx(t)\,, \qquad (2.2.36b)$$

$$x(0) = x_0 \qquad (2.2.36c)$$

which is assumed to be reachable and observable, consider the performance index

$$J = \frac{1}{2}\int_0^\infty [y'(t)y(t) + \rho u'(t)Ru(t)]dt \qquad (2.2.37)$$

where $R = R' > 0$ is given and $\rho > 0$ is a scalar. The desired behaviour for the system is $y(\cdot) = 0$.
Under the assumptions above, the solution of the Problem 2.2.3 defined by equations (2.2.36)-(2.2.37) exists for each initial state and $\rho > 0$ and is specified by the control law

$$u_{cs}^o(x,\rho) = -\frac{1}{\rho}R^{-1}B'P(\rho)x \qquad (2.2.38)$$

where $P(\rho)$ is the (unique) positive definite solution of the ARE

$$0 = PA + A'P - \frac{1}{\rho}PBR^{-1}B'P + C'C .$$

A preliminary result concerning the asymptotic properties of $P(\rho)$, is given in the following lemma where $Q := C'C$, $x^o(t,\rho)$ is the solution of equations (2.2.36a), (2.2.36c),(2.2.38) and $u^o(t,\rho)$ is the relevant control.

Lemma 2.2.4 The limit of $P(\rho)$ as $\rho \to 0^+$ exists and is denoted by P_0.
A meaningful measure of how similar the system response is to the desired one (that is how $y(\cdot)$ is closed to zero) is supplied by the quantity

$$J_x(x_0,\rho) := \frac{1}{2}\int_0^\infty x^o{}'(t,\rho)Qx^o(t,\rho)dt$$

the limiting value of which is given in the following theorem.

Theorem 2.2.11 Let P_0 be the limit value of $P(\rho)$. Then

$$\lim_{\rho\to 0^+} J_x(x_0,\rho) = \frac{1}{2}x_0{}'P_0x_0 .$$

Thanks to this theorem matrix P_0 supplies the required information about the maximum achievable accuracy in attaining the desired behaviour ($y(\cdot) = 0$). The best result is $P_0 = 0$. The circumstances under which this happens are specified in the forthcoming theorem. For the sake of simplicity, matrices *B* and *C* (which appear in equations (2.2.36a), (2.2.36b) and have dimensions $n\times m$ and $p\times n$) are assumed to have rank equal to *m* and *p*, respectively.

Theorem 2.2.12 Let system (2.2.36a), (2.2.36b) be both reachable and observable. The following conclusions hold:

i) If $m < p$, then $P_0 \neq 0$;

ii) If $m = p$, then $P_0 = 0$ if and only if the transmission zeros of the system have nonpositive real parts;

iii) If $m > p$ and there exists a full rank $m\times p$ matrix *M* such that the transmission zeros of system $\Sigma(A, BM, C, 0)$ have nonpositive real parts, then $P_0 = 0$.

2.2.4.4 Inverse problem

The ***inverse*** optimal control problem consists in finding, for a given system and ***control law***, a performance index with respect to which such a control law is optimal. The discussion as well as the solution of this seemingly useless problem allows us to clarify the ultimate properties of a control law in order that it can be considered optimal and to precisely evaluate the number of degrees of freedom, which are actually available when designing a control law via an LQ approach.
We will deal only with the case of scalar control variable and time invariant system and control law. Thus, the inverse problem is stated on the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.2.39}$$

and the control law

$$u(x) = Kx \tag{2.2.40}$$

where the control variable u is scalar, while the state vector x has n components. The problem consists in finding a matrix $Q = Q' \geq 0$ such that the control law (2.2.40) is optimal relative to the system (2.2.39) and the performance index

$$J = \int_0^\infty [x'(t)Qx(t) + u^2(t)]dt\,. \tag{2.2.41}$$

Since u is scalar, there is no lack of generality in taking $R = 1$.
For the above problem a few results are available, among which the most significant one is stated in the following theorem where K is assumed to be nonzero. If this is not the case, the solution of the problem would be trivial, namely, seemingly $Q = 0$.

Theorem 2.2.13 With the reference to the system (2.2.39) and the control law (2.2.40) where $K \neq 0$, let the following assumptions be assumed:

(a1) The pair (A, B) is reachable;
(a2) The system (2.2.39),(2.2.40) is asymptotically stable;
(a3) $\mu(j\omega) := \left|1 - K(j\omega I - A)^{-1}B\right| \geq 1,\ \forall\omega$ real, $\mu(\cdot) \neq 1$;
(a4) The pair (A, K) is observable.

Then there exists $Q = Q' \geq 0$ such that the control law (2.2.40) is optimal for the LQ problem defined on the system (2.2.39) and the performance index (2.2.41).

Remark 2.2.20 (*Unnecessity of assumption* (a4)) The forth assumption in Theorem can be removed. Indeed, let the pair (A, K) be not observable and its unobservable part be already put into evidence, so that

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 & 0 \end{bmatrix}$$

with the pair (A_1, K_1) observable and the pair (A_1, B_1) reachable (this last claim follows from the reachability assumption of the pair (A, B)). Theorem 2.2.13 can be applied to the subsystem Σ_1 described by $\dot{x}_o = A_1 x_o + B_1 u$ and the control law $u = K_1 x_o$. In fact, assumption (a2) is verified for the triple (A_1, B_1, K_1) if it holds for the triple (A, B, K), since

$$A+BK=\begin{bmatrix} A_1+B_1K_1 & 0 \\ A_2+B_2K_1 & A_3 \end{bmatrix}.$$

Furthermore, since

$$K(sI-A)^{-1}B=K_1(sI-A_1)^{-1}B_1,$$

if condition (a3) holds for the triple (A,B,K) it holds also for the triple (A_1,B_1,K_1). Thus a matrix Q_1 can be found which defines a performance index relative to which K_1x is an optimal control law for the subsystem Σ_1. It is then obvious that matrix $Q=diag[Q_1 \quad 0]$ specifies a performance index corresponding to which the given control law is optimal (the state of the subsystem $\dot{x}_{no}=A_2x_o+A_3x_{no}+B_2u$ affects neither in a direct nor in an indirect way the performance index. Thus, it should not contribute to the current value of the control variable).

Remark 2.2.21 (*Degrees of freedom in the choice of the performance index*)
Only the structure of the performance index should be considered as given. The relevant free parameters are being selected (through a sequence of rationally performed trails) so as to specify a satisfactory control law. If the control law is scalar, the number of theses design parameters is substantially less than $1+\dfrac{n(n+1)}{2}$, that is the number of elements in *R* and *Q*. On one hand, *R* can be set to 1 without loss of generality, while, on the other hand, under the mild assumptions of reachability of (A,B) and stability of the feedback system, conditions (a1)-(a3) of Theorem 2.2.13 are satisfied whenever the control law results from the solution of the LQ problem corresponding to an arbitrary $Q\geq 0$. These conditions are sufficient to ensure the existence of the solution of the inverse problem. Thus, the same control law must also result from a *Q* expressed as the product of an *n*-vector by its transpose and the really free parameters are *n*.

Conditions (a1)-(a3) can further be weakened, up to becoming necessary and sufficient. These new conditions are presented in the following theorem where the triple (A,B,K) has already undergone the canonical decomposition, thus exhibiting the form

$$A=\begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_5 & 0 & A_6 \\ 0 & 0 & A_7 & A_8 \\ 0 & 0 & 0 & A_9 \end{bmatrix}, \qquad B=\begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \quad K=[0 \quad K_1 \quad 0 \quad K_2] \tag{2.247}$$

with the pair (A_r,B_r) reachable, the pair (A_o,K_o) observable and

$$A_r:=\begin{bmatrix} A_1 & A_2 \\ 0 & A_5 \end{bmatrix}, \quad B_r:=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$A_o:=\begin{bmatrix} A_5 & A_6 \\ 0 & A_9 \end{bmatrix}, \quad K_o:=[K_1 \quad K_2].$$

Theorem 2.2.14 With reference to the system (2.2.39) and the control law (2.2.40) there exists a matrix $Q=Q'\geq 0$ such that the optimal regulator problem defined on that system and the performance index (2.2.41) admits a solution for each initial state. The solutions specified by the given control law if and only if

(a1) All the eigenvalues of A_9 have negative real part,

(a2) All the eigenvalues of $A_5 + B_2 K_1$ have negative real part,

(a3) One of the two following conditions holds:
(a31) $K = 0$
(a32) $\mu(j\omega) := \left|1 - K_1 (j\omega I - A_5)^{-1} B_2\right| \geq 1, \ \ \forall \omega \ \ real, \ \ \mu(\cdot) \neq 1$.

2.3 The LQG problem

2.3.1 Introduction

The discussion in this chapter is dedicated to a not purely deterministic framework and focused on two problems, which concern with one and the same stochastic system. The connection between these problems and the previously presented material is not apparent from the very beginning while it will be shown to be very tight. Reference is made to a stochastic system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t) \tag{2.3.1a}$$

$$y(t) = C(t)x(t) + w(t) \tag{2.3.1b}$$

$$x(t_0) = x_0 \tag{2.3.1c}$$

where, as customary, A, B, C are continuously differentiable functions. In equations (2.3.1a), (2.3.1b) $[v' \quad w']'$ is a ***zero mean, gaussian stationary,*** $(n+p)$-dimensional ***stochastic process*** (n and p are the dimensions of the state and output vectors) which is assumed to be a while noise. In equation (2.3.1c) the initial state is an n-dimensional ***gaussian*** random variable independent from $[v' \quad w']'$. The uncertainty on system (2.3.1) is thus specified by

$$E[\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}] = 0, \ \ \forall t, \tag{2.3.2}$$

$$E[x_0] = \bar{x}_0, \tag{2.3.3}$$

and

$$E[\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}[v'(t) \quad w'(t)]] = \begin{bmatrix} V & Z \\ Z' & W \end{bmatrix}\delta(t-\tau) := \Xi\delta(t-\tau), \quad \forall t, \tau, \tag{2.3.4}$$

$$E[x_0[v'(t) \quad w'(t)]] = 0, \quad \forall t, \tag{2.3.5}$$

$$E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = \Pi_0, \tag{2.3.6}$$

where the quantities $\bar{x}_0$, V, Z, W, Π_0 are given and δ is the impulsive function. Moreover, matrices V, W, Ξ, Π_0 are symmetric and positive semidefinite.

The two problems under consideration are concerned with the optimal ***estimate*** of the state of system (2.3.1) and its optimal (***stochastic***) control. The first problem is to determine the optimal approximation $\hat{x}(t_f)$ of $\hat{x}(t_f)$, relying on all available information, namely the time history of the control and output variables (*u* and *y*) on the interval $[t_0, t_f]$ and the uncertainty characterization provided by equations (2.3.2)-(2.3.6). The second problem is to design a regulator with input *y*, which generates the control *u* so as to minimize a suitable performance criterion.

Remark 2.3.1 (*Different system models*)
When the system under consideration is described by the equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + B^* v^*(t) \tag{2.3.7}$$

where v^* is a zero mean white noise independent from x_0 and characterized by

$$E[\begin{bmatrix} v^*(t) \\ w(t) \end{bmatrix}[v^{*\prime}(t) \quad w'(t)]] := \Xi\delta(t-\tau)$$

with

$$\Xi' = \Xi = \begin{bmatrix} V^* & Z^* \\ Z^{*\prime} & W \end{bmatrix} \geq 0, \quad V^* > 0.$$

In this case the previously presented formulation is still adopted by defining the stochastic process $v := B^* v^*$ which verifies equations (2.3.1a), (2.3.2), (2.3.4), (2.3.5) with $V := B^* V^* B^{*\prime}$, $Z := B^* Z^*$.
At other times, equations (2.3.1a), (2.3.1b) are replaced by equations (2.3.7) and

$$y(t) = C(t)x(t) + C^* v^*(t) \tag{2.3.8}$$

where v^* is a zero mean white noise independent of x_0 with intensity $V^* > 0$. Letting $v := B^* v^*$ and $w := C^* v^*$, it is straightforward to get back to equations (2.3.1)-(2.3.6) with $V := B^* V^* B^{*\prime}$, $Z := B^* Z^*$, $W := C^* V^* C^{*\prime}$.

2.3.2 Kalman filter

The problem of the optimal estimate or filtering of the state of system (2.3.1)-(2.3.6) is considered in this section. The adopted performance criterion for the performed estimate is the expected value of the square of the error undergone in evaluating an arbitrarily given linear combination of the state components. Thus the problem to be discussed can be formally described as follows.

Problem 2.3.1 (*Optimal estimate of* $b'x(t_f)$) Given an arbitrary vector $b \in R^n$, determine, on the basis of the knowledge of $y(t)$ and $u(t)$, $t_0 \leq t \leq t_f$, a scalar β such that the quantity

$$J_b := E[(b'x(t_f) - \beta)^2] \tag{2.3.9}$$

is minimized with reference to the system (2.3.1)-(2.3.6).

We will consider two cases. The first one is the ***nominal*** case where the matrix *W* (the intensity of the output noise *w*) is positive definite. The second one is the singular case where *W* is positive semidefinite. These two cases differ significantly from each other not only from technical point of view, but also of the meaning of the underlying problem. In order to understand this let recall that the matrix W is symmetric and it can be presented as

$$W := T'DT$$

where T is an orthogonal matrix, matrix D is diagonal $D := diag[d_1, d_2, \ldots, d_r, 0, \ldots, 0]$ and $rank(W) = r$. By letting

$$y^* := Ty\,,$$

it follows that

$$y^* = TCx + Tw := C^*x + w^*$$

with

$$E[w^*(t)] = 0 \text{ and } E[w^*(t)w^{*\prime}(\tau)] = TE[w(t)w'(\tau)]T' = D\,,$$

so that it is possible to conclude that the last $p - r$ components of y^* are not corrupted by noise. In other words, the assumption $W > 0$ gives that no outputs or their linear combinations are noise free.

2.3.2.1 Nominal case

The optimal state estimation problem is now considered under the assumption that the matrix W is positive definite. First, the observation interval is supposed to be finite. Subsequently, the case of infinite interval will be tackled.

Let $-\infty < t_0 < t_f < \infty$. The Problem 2.3.1 is discussed under the additional constraint to the scalar β, which is asked to ***linearly*** depend on y, according to the equation

$$\beta = \int_{t_0}^{t_f} \vartheta'(t)y(t)dt \tag{2.3.10}$$

where the function $\vartheta(t)$ must be selected so as to minimize the value of the criterion (2.3.9). However, it is possible to prove that the choice (2.3.100 for the form of the estimate of $b'x(t_f)$ does not actually cause any loss in optimality, since in the adopted stochastic framework the estimate which minimizes J_b is indeed of that form. With the reference to the selection of ϑ the following result holds.

Theorem 2.3.1 Consider equation (2.3.10). The function ϑ^0 which solves Problem 2.3.1 relative to system (2.3.1)-(2.3.6) when the observation interval is finite, $u(\cdot) = 0$, $\bar{x}_0 = 0$ and $Z = 0$, is given by

$$\vartheta^0(t) = W^{-1}C(t)\Pi(t)\alpha^0(t) \tag{2.3.11}$$

where Π is the (unique, symmetric, positive semidefinite) solution of the differential Riccati equation

$$\dot{\Pi}(t) = \Pi(t)A'(t) + A(t)\Pi(t) - \Pi(t)C'(t)W^{-1}C(t)\Pi(t) + V(t) \tag{2.3.12}$$

satisfying the boundary condition

$$\Pi(t_0) = \Pi_0 \tag{2.3.13}$$

while α^0 is the unique solution of the linear equation

$$\dot{\alpha}(t) = -[A(t) - \Pi(t)C'(t)W^{-1}C(t)]'\alpha(t) \tag{2.3.14}$$

satisfying the boundary condition

$$\alpha(t_f) = b \tag{2.3.15}$$

Some explanations: if consider, over the interval $[t_0, t_f]$, the dynamical system

$$\dot{\alpha} = -A'\alpha + C'\vartheta \tag{2.3.16a}$$

$$\alpha(t_f) = b \tag{2.3.16b}$$

in view of equations (2.3.1a),(2.3.1b) it follows that

$$\frac{d(\alpha' x)}{dt} = \vartheta' y - \vartheta' w + \alpha' v$$

By integrating both sides of this equation between t_0 and t_f we get in view of equations (2.3.10),(2.3.15),

$$b'(t_f) - \beta = \alpha'(t_0)x(t_0) - \int_{t_0}^{t_f} \vartheta'(t)w(t)dt + \int_{t_0}^{t_f} \alpha'(t)v(t)dt$$

By squaring both sides of this equation, performing the expected value operation, exploiting the linearity of the operator E and the identity $(r's)^2 = r'ss'r$ and taking into account equation (2.3.2)-(2.3.6) it follows that

$$J_b = \alpha'(t_0)\Pi_0\alpha(t_0) + \int_{t_0}^{t_f} [\alpha'(t)V\alpha(t) + \vartheta'(t)W\vartheta(t)]dt \tag{2.3.17}$$

and selecting ϑ^0 so as to minimize J_b amounts to solving the optimal control problem defined by the linear system (2.3.16) and the quadratic performance index (2,3,17), i.e. an LQ problem where the roles of the final and initial times have been interchanged.

Remark 2.3.2 (*Meaning of* β^0)
Within the particular framework into which Theorem2.3.1 is embedded, both $x(t)$ and $y(t)$ are zero mean random variables because $\bar{x}_0 = 0$ and v and w are zero mean white noises. Therefore β^0 is a zero mean random variable as well and its value, as given in Theorem 2.3.1, is the one which minimizes the variance of the estimation error of $b'x(t_f)$.

Remark 2.3.3 (*Variance of the estimation error*)
Theorem 2.3.1 allows us to easily conclude that the optimal value of the performance criterion is

$$J_b^o = b'\Pi(t_f)b$$

which is the minimal variance of the estimation error at time t_f. Thus, the variance depends on the value of the matrix Π at that time. Note that the final time t_f and the initial time t_0

are finite and given but generic. Therefore $b'\Pi(t)b$ is the minimal variance of the estimation error of $b'x(t)$.

Remark 2.3.4 (*Correlated noises*)
When *v* and *w* are correlated noises $Z \neq 0$. In fact, it is easy to check that equation (2.3.17) becomes

$$J_b = \alpha'(t_0)\Pi_0\alpha(t_0) + \int_{t_0}^{t_f}[\alpha'(t)V\alpha(t) - 2\alpha'(t)Z\vartheta(t) + \vartheta'(t)W\vartheta(t)]dt$$

so that the estimation problem reduces to an LQ problem with a rectangular term which can be managed as shown in Remark 2.2.4. However, observe that matrix $V_c := V - ZW^{-1}Z'$ is positive semidefinite, since $V_c = T\Xi T'$ where $T := \begin{bmatrix} I & -ZW^{-1} \end{bmatrix}$ and, from equation (2.3.4), $\Xi \geq 0$. Thus, Theorem 2.3.1 holds with *V*, $A(t)$ replaced by V_c, $A_c(t) := A(t) - ZW^{-1}C(t)$, respectively, and equation (2.3.11) replaced by

$$\vartheta^o(t) = W^{-1}(C(t)\Pi(t) + Z')\alpha^o(t).$$

In view of this discussion there is no true loss of generality in considering only the case $Z = 0$.
The importance of Theorem 2.3.1 is grater than might appear from its statement. Indeed, it allows us to devise the structure of a dynamic system the state of which, $\hat{x}(t)$ is, for each $t \in [t_0, t_f]$, the optimal estimate of the state of (2.3.1)-(2.3.6). This fact is presented in the next Theorem.

Theorem 2.3.2 Consider the system (2.3.1)—(2.3.6) whit $u(\cdot) = 0$, $\bar{x}_0 = 0$ and $Z = 0$. Then, for each $b \in R^n$ and for $-\infty < t_0 \leq t \leq t_f < \infty$ the optimal estimate of $b'x(t)$ is $b'\hat{x}(t)$, $\hat{x}(t)$ being the state, at time *t*, of the system

$$\dot{\hat{x}}(t) = [A(t) + L(t)C(t)]\hat{x}(t) - L(t)y(t), \quad (2.3.18a)$$

$$\hat{x}(t_0) = 0 \quad (2.3.18b)$$

where $L(t) := -\Pi(t)C'(t)W^{-1}$ and Π is the solution (unique, symmetric and positive semidefinite) of the differential Riccati equation (2.3.12).

The above results can easily be generalized to cope with the case where $u(\cdot) \neq 0$ and $\bar{x}_0 \neq 0$ since the linearity of the system allows us to independently evaluate the effects of *x* of the deterministic input *u* and the time propagation of the expected value of the initial state. The presence of the deterministic input is taken into account by simply adding the term Bu to the equation of $\dot{\hat{x}}$, while the propagation of the state expected value is correctly performed by giving the value $\bar{x}_0$ to $\hat{x}(t_0)$.
Recalling that $E[\hat{x}(t_0) - x(t_0)] = 0$, we can conclude that, $E[\hat{x}(t) - x(t)] = 0, \ \forall t$, and $b'\hat{x}(t)$ is still the estimate of $b'x(t)$ which entails an error with minimal variance. In short, $b'\hat{x}(t)$ is said to be the optimal or *minimal variance* estimate of $b'x(t)$, thus justifying the commonly adopted terminology according to which Problem 2.3.1 is the minimal variance estimation problem. These remarks are collected in the following theorem.

Theorem 2.3.3 Consider the system (2.3.1)—(2.3.6) whit $Z = 0$. Then, for each $b \in R^n$ and for $-\infty < t_0 \le t \le t_f < \infty$ the optimal variance estimate of $b' x(t)$ is $b' \hat{x}(t)$, $\hat{x}(t)$ being the state, at time *t*, of the system

$$\dot{\hat{x}}(t) = [A(t) + L(t)C(t)]\hat{x}(t) - L(t)y(t) + B(t)u(t), \qquad (2.3.19a)$$

$$\hat{x}(t_0) = \bar{x}_0 \qquad (2.3.19b)$$

where $L(t) := -\Pi(t)C'(t)W^{-1}$ and Π is the solution (unique, symmetric and positive semidefinite) of the differential Riccati equation (2.3.12) satisfying the boundary condition (2.3.13).

Remark 2.3.5 *(Not strictly proper system)*
In view of the discussion preceding Theorem 2.3.3, it is quite obvious how the case where a term $D(t)u(t)$ appears in eq. (2.3.1b) can be handed. Indeed, it is sufficient to add to $\hat{y}$, which is the optimal estimate of *y*, the term Du, so that equation (2.3.19a) becomes

$$\dot{\hat{x}} = (A + LC)\hat{x} - Ly + (B + LD)u \; .$$

Remark 2.3.6 *(Meaning of $\Pi(t)$)*
By referring to the proof of Theorem 2.3.1, it is easy to conclude that

$$b'\Pi(t)b = E[(b' x(t) - b' \hat{x}(t))^2] = b' E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))']b \; .$$

Since *b* is arbitrary, the matrix $\Pi(t)$ is the variance of the optimal estimation error at time *t*: therefore, any norm of it, for instance its *trace*, constitutes, when evaluated at some time τ, a meaningful measure of how good is the estimate performed on the bases of the data available up to τ.

Remark 2.3.7 *(Incorrelation between the estimation error and the filter state)*
An interesting property of the Kalman filter is put into evidence by the following discussion. Let $e := x - \hat{x}$ and consider the system with state $[e' \quad \hat{x}']$, which is deduced by the equations

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + LC & 0 \\ -LC & A \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} + \begin{bmatrix} I & L \\ 0 & -L \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u \; .$$

By denoting with $\bar{e}(t)$ and $\bar{\hat{x}}(t)$ the expected values of $e(t)$ and $\hat{x}(t)$, respectively, and letting

$$E[\begin{bmatrix} e(t) - \bar{e}(t) \\ \hat{x}(t) - \bar{\hat{x}}(t) \end{bmatrix} [e'(t) - \bar{e}'(t) \quad \hat{x}'(t) - \bar{\hat{x}}'(t)]] := \begin{bmatrix} \Pi_{11}(t) & \Pi_{12}(t) \\ \Pi_{12}'(t) & \Pi_{22}(t) \end{bmatrix},$$

it follows that the matrices Π_{ij}, $i = 1,2;\quad j = 1,2$, satisfy the differential equations

$$\dot{\Pi}_{11} = \Pi_{11}(A + LC)' + (A + LC)\Pi_{11} + V + LWL' \qquad (2.3.20)$$

$$\dot{\Pi}_{12} = (A + LC)\Pi_{12} + \Pi_{12}A' - \Pi_{11}C'L' - LWL' \qquad (2.3.21)$$

$$\dot{\Pi}_{22} = -LC\Pi_{12} - \Pi_{12}'C'L' + A\Pi_{22} + \Pi_{22}A' + LWL'$$

with the boundary conditions

$$\Pi_{11}(t_0) = \Pi_0, \tag{2.3.22}$$

$$\Pi_{12}(t_0) = 0, \tag{2.3.23}$$

$$\Pi_{22}(t_0) = 0 .$$

By recalling that $L = -\Pi C'W^{-1}$, it is straightforward to check that equation (2.3.20) coincides with equation(2.3.12) and that equation (2.3.22) is indeed equation (2.3.13), so that $\Pi_{11}(\cdot) = \Pi(\cdot)$. From this identity, it follows that in equation (2.3.21) it is $-\Pi_{11}C'L' - LWL' = 0$: thus, $\Pi_{12}(\cdot) = 0$ solves such an equation with the relevant boundary condition (2.3.23). This fact proves that the stochastic processes *e* and $\dot{x}$ are uncorrelated. By exploiting Remark 2.3.4, the same arguments can be extended to the case where *v* and *w* are correlated $(Z \neq 0)$.

The proof of Theorem 2.3.1 suggests which results pertaining to LQ problems are useful in the case of an unbounded observation interval, that is when $t_0 = -\infty$. By referring to Section 2.2.3 of Chapter 2.2, the initial state of the system is supposed to be known and equal to zero, so that $\bar{x}_0 = 0$, $\Pi_0 = 0$ and a suitable *reconstructability* assumption is introduced (recall that this property is dual to controllability). On this basis, the forthcoming theorem can be stated.

Theorem 2.3.4 Let the pair $(A(t), C(t))$ be reconstructable for $t \le t_f$. Then the problem of the optimal state estimation for the system (2.3.1)—(2.3.6) with $Z = 0$, $\bar{x}_0 = 0$, $\Pi_0 = 0$ admits a solution also when $t_0 = -\infty$. For each $b \in R^n$ and $\tau \le t_f$ the optimal estimate of $b'x(\tau)$ is given by $b'\hat{x}_\infty(\tau)$, where $\hat{x}_\infty(\tau)$ is the limit approached by the solution, evaluated at τ, of the equation

$$\dot{\hat{x}}(t) = [A(t) + L(t)C(t)]\hat{x}(t) - L(t)y(t) + Bu(t), \tag{2.3.24}$$

$$\bar{x}(t_0) = 0$$

when $t_0 = -\infty$. In equation (2.3.24) $L(t) := -\overline{\Pi}(t)C'(t)W^{-1}$ and, for all *t*, $\overline{\Pi}$ is a symmetric and positive semidefinite matrix given by

$$\overline{\Pi}(t) = \lim_{t_0 \to -\infty} \Pi(t, t_0),$$

$\Pi(t, t_0)$ being the solution *(unique, symmetric and positive semidefinite)* of the differential Riccati equation (2.3.12) satisfying the boundary condition $\Pi(t_0, t_0) = 0$.

Thus, the apparatus, which supplies the optimal estimate possesses the structure shown in in Fig. 2.3.2 (with $\bar{x}_0 = 0$ and, if it is the case, the term Du added to $\hat{y}$) also when the observation interval is unbounded.

In a similar way, it is straightforward to handle filtering problems over an unbounded observation interval when the system is time-invariant: indeed, it is sufficient to mimic the results relevant to the optimal regulator problem in order to state the following theorem which refers to the time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t), \tag{2.3.25a}$$

$$y(t) = Cx(t) + w(t). \tag{2.3.25b}$$

Theorem 2.3.5 Consider the system (2.3.25), (2.3.1c)—(2.3.6) with $Z=0$, $\bar{x}_0=0$, $\Pi_0=0$ and the pair (A,C) observable. Then the problem of the optimal state estimation admits a solution also when $t_0 \to -\infty$. For each $b \in R^n$ and $\tau \le t_f$ the optimal estimate of $b'x(\tau)$ is given by $b'\hat{x}_\infty(\tau)$, where $\hat{x}_\infty(\tau)$ is the limit approached by the solution, evaluated at τ, of the equation

$$\dot{\hat{x}}(t)=[A+\bar{L}C]\hat{x}(t)-\bar{L}y(t)+Bu(t), \tag{2.3.26}$$

$$\bar{x}(t_0)=0$$

when $t_0=-\infty$. In equ. (2.3.26) $\bar{L}:=-\bar{\Pi}C'W^{-1}$, $\bar{\Pi}$ being a constant matrix, symmetric and positive semidefinite, which solves the algebraic Riccati equation

$$0=\Pi A'+A\Pi-\Pi C'W^{-1}C\Pi+V \tag{2.3.27}$$

and is such that

$$\bar{\Pi}(t)=\lim_{t_0\to-\infty}\Pi(t,t_0),$$

$\Pi(t,t_0)$ being the solution *(unique, symmetric and positive semidefinite)* of the differential Riccati equation (2.3.12) satisfying the boundary condition $\Pi(t_0,t_0)=0$.

Obviously, then Theorem 2.3.5 applies the Kalman filter is a time-invariant system, the stability properties of which can be analyzed as done within the framework of the optimal regulator problem (Section 2.2.4.1 of Chapter 2.2.4). All the results there are still valid, provided the necessary modifications have been brought. As an example, the particularly meaningful result concerning asymptotic stability can be stated as shown in the following theorem.

Theorem 2.3.6 Consider the system (2.2.25) and let the triple (A,F,C) be minimal, F' being any factorization of *V*. Then the Kalman filter relevant to an unbounded observation interval is asymptotically stable, i.e. all the eigenvalues of the matrix $A+\bar{L}C$ have negative real parts.

2.3.2.2 Singular case

A possible way to dealing with the filtering problem in the singular case is now presented with reference to the time-invariant system described by (2.3.25). Thus the intensity of the output noise is a matrix *W* which is not positive definite, i.e. $W\ge 0$, $\det W=0$ and , for the sake of simplicity, the rank of matrix *C* (2.3.25b) is assumed to be equal to the number *p* of its rows.

Denote with $T:=[T_1' \quad T_2']$ an orthogonal matrix such that

$$TWT'=\begin{bmatrix}\Omega & 0\\ 0 & 0\end{bmatrix}$$

where Ω is a nonsingular matrix of dimensions $p_1<p$. Letting $y^*(t):=Ty(t)$, it follows that

$$y^*(t):=\begin{bmatrix}y_d(t)\\ y_c(t)\end{bmatrix}=\begin{bmatrix}T_1Cx(t)\\ T_2Cx(t)\end{bmatrix}+\begin{bmatrix}T_1w(t)\\ T_2w(t)\end{bmatrix}$$

In view of the fact that the intensity of the white noise $T_2 w$ is zero, this relation can be rewritten as

$$y_d(t) = C_d x(t) + w^*(t)\,, \tag{2.3.28a}$$

$$y_c(t) = C_c x(t) \tag{2.3.28b}$$

where $C_d := T_1 C$, $C_c := T_2 C$ and $w^*(t) := T_1 w(t)$. The vector u_d (with p_1 components) is thus constructed by those output variables, which are actually affected by the noise *w* while the vector y_c (with $p - p_1$ components) accounts for those output variables, which are not affected by it. Therefore, the noise-free information carried by y_c should be exploited in tackling the state estimation problem. In this regard, let C^* be an $n-(p-p_1)\times n$ matrix such that $[C_c' \quad C^*]'$ is nonsingular and denote with $[\Gamma_c \quad \Gamma^*]$ the inverse of this last matrix. It follows that

$$x(t) = \Gamma_c y_c(t) + \Gamma^* x^{(1)}(t) \tag{2.3.29}$$

where

$$x^{(1)}(t) := C^* x(t)\,. \tag{2.3.30}$$

In principal, the time derivative of the noise-free function y_c can be computed so that from eqs. (2.3.25a), (2.3.16a), (2,3,28)—(2.3.30) it follows that

$$\begin{aligned}\dot{y}_c(t) &= C_c[Ax(t) + Bu(t) + v(t)] = \\ &= C_c A\Gamma_c y_c(t) + C_c A\Gamma^* x^{(1)}(t) + C_c Bu(t) + C_c v(t)\end{aligned} \tag{2.3.31a}$$

$$\begin{aligned}\dot{x}^{(1)}(t) &= C^*[Ax(t) + Bu(t) + v(t)] = \\ &= C^* A\Gamma_c y_c(t) + C^* A\Gamma^* x^{(1)}(t) + C^* Bu(t) + C^* v(t)\end{aligned} \tag{2.3.31b}$$

$$y_d(t) = C_d \Gamma_c y_c(t) + C_d \Gamma^* x^{(1)}(t) + w^*(t) \tag{2.3.31c}$$

Equations (2.3.31) define a dynamic system with state $x^{(1)}$, known inputs *u* and y_c, unknown inputs (noises) *v* and w^*, outputs y_d and $\dot{y}_c$. More concisely, equations (2.3.31) become

$$\dot{x}^{(1)}(t) = A^{(1)} x^{(1)}(t) + B^{(1)} u^{(1)}(t) + v^{(1)}(t), \tag{2.3.32a}$$

$$y^{(1)}(t) = C^{(1)} x^{(1)}(t) + D^{(1)} u^{(1)}(t) + w^{(1)}(t), \tag{2.3.32b}$$

where $v^{(1)}(t) := C^* v(t)$, $u^{(1)}(t) := \begin{bmatrix} y_c(t) \\ u(t) \end{bmatrix}$, $y^{(1)}(t) := \begin{bmatrix} y_d(t) \\ \dot{y}_c(t) \end{bmatrix}$, $w^{(1)}(t) := \begin{bmatrix} w^*(t) \\ C_c v(t) \end{bmatrix}$

and $A^{(1)} := C^* A \Gamma^*$, $B^{(1)} := [C^* A\Gamma_c \quad C^* B]$, $C^{(1)} := \begin{bmatrix} C_d \Gamma^* \\ C_c A \Gamma^* \end{bmatrix}$, $D^{(1)} := \begin{bmatrix} C_d \Gamma_c & 0 \\ C_c A \Gamma_c & C_c B \end{bmatrix}$.

The intensity of the white noise $[v^{(1)}{}' \quad w^{(1)}{}']$ is

$$\Xi^{(1)} := \begin{bmatrix} C^* V C^*{}' & C^* Z T_1' & C^* V C_c' \\ T_1 Z' C^*{}' & \Omega & T_1 Z' C_c' \\ C_c V C^*{}' & C_c Z T_1 & C_c V C_c' \end{bmatrix}.$$

If the intensity of $w^{(1)}$, i.e. the matrix

$$W^{(1)} := \begin{bmatrix} \Omega & T_1 Z' C_c' \\ C_c Z T_1 & C_c V C_c' \end{bmatrix}$$

is positive definite, the filtering problem relative to the system (2.3.32) is normal and the results of the preceding section can be applied, provided that the probabilistic characterization of $x^{(1)}(t_0)$ could be performed on the basis of *all* available information. If, on the other hand, $W^{(1)}$ is not positive definite, the above outlined procedure can again be applied. Note that the dimension of the vector $x^{(1)}$ is strictly less than *n*. Thus, the procedure can be iterated only a finite number of times, and either the situation is reached where $W^{(1)} > 0$ or *n* noise-free outputs are available and suited to exactly estimated *x*.

Assuming, for the sake of simplicity, that $W^{(1)}$ is positive definite, the expected value and variance of the initial state of system (2.3.32a) has to be computed, given $y_c(t_0) = C_c x(t_0)$. If $x(t_0)$ is not a *degenerate* gaussian random variable, i.e. if $\Pi_0 > 0$, it can be shown that the following relation hold

$$\bar{x}_0^{(1)} := E[x^{(1)}(t_0) y_c(t_0)] = C^*[\bar{x}_0 + \Pi_0 C_c'(C_c \Pi_0 C_c')^{-1}[y_c(t_0) - C_c \bar{x}_0]],$$

$$\Pi_0^{(1)} := E[(x^{(1)}(t_0) - \bar{x}_0^{(1)})(x^{(1)}(t_0) - \bar{x}_0^{(1)})' | y_c(t_0)] = C^*[\Pi_0 + \Pi_0 C_c'(C_c \Pi_0 C_c')^{-1} C_c \Pi_0] C^{*\prime}.$$

Let $\Pi^{(1)}$ be the solution (unique, symmetric and positive semidefinite) of the DRE

$$\dot{\Pi}^{(1)}(t) := A_c^{(1)} \Pi^{(1)}(t) + \Pi^{(1)}(t) A_c^{(1)\prime} + V_c^{(1)} - \Pi^{(1)}(t) C^{(1)\prime} (W^{(1)})^{-1} C^{(1)} \Pi^{(1)}(t)$$

satisfying the boundary condition $\Pi^{(1)}(t_0) = \Pi_0^{(1)}$. In this equation

$$A_c^{(1)} := A^{(1)} - Z^{(1)}(W^{(1)})^{-1} C^{(1)}, \; Z^{(1)} := \begin{bmatrix} C^* Z T_1' & C^* V C - C' \end{bmatrix},$$

$$V_c^{(1)} := V^{(1)} - Z^{(1)} (W^{(1)})^{-1} Z^{(1)\prime} \text{ and } V^{(1)} := C^* V C^{*\prime}.$$

Then the Kalman filter for the system (2.3.32) when the uncertainty is specified as above and the observation interval is finite, possesses the *gains* L_c and L_d given by the equation

$$\begin{bmatrix} L_d(t) & L_c(t) \end{bmatrix} = -[\Pi^{(1)}(t) C^{(1)\prime} + Z^{(1)}](W^{(1)})^{-1}. \qquad (2.3.33)$$

Notice that these gains are time-varying.

The actual implementation of the last result does not need differentiation. In fact, the signal y_c, after differentiation and multiplication by $-L_c$, undergoes integration. Since $\Pi^{(1)}$ is a differentiable function, from equation (2.3.33) it follows that also the function L_c is such, so that

$$\int L_c(t) \dot{y}_c(t) dt = L_c(t) y_c(t) - \int \dot{L}_c(t) y_c(t) dt$$

and the filter can be implemented without differentiation since $\dot{L}_c$ can be evaluated in advance.

2.3.3 LQG control problem

The optimal control problem to be considered in this section refers to the system (2.3.1)-(2.3.6) and the performance index

$$J = E\left[\frac{1}{t_f - t_0}\int_{t_0}^{t_f}(x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt\right] \tag{2.3.34}$$

where, as in the LQ context, $Q(t) = Q'(t) \geq 0$ and $R(t) = R'(t) > 0,\ \forall t$, are matrices of continuously differentiable functions and $Q(\cdot) \neq 0$ to avoid triviality. In the performance index (2.3.34) a term, which is a quadratic nonnegative function of $x(t_f)$ could also be added. Its presence, however, does not after the essence of the problem but rather makes the discussion a little more involved. For the sake of simplicity here and likewise the intensity *W* of the noise *w* is assumed to be positive definite.

The Linear Quadratic Gaussian (LQG) optimal control problem under consideration is defined in the following way

Theorem 2.3.2 (LQG problem) Consider the system (2.3.1)—(2.3.6): Find the control, which minimizes the performance index (2.3.34).

In problem 2.3.2, the control interval is given and may or may not be finite. In the first case, it is obvious that the multiplicative factor in front of the integral is not important, while in the second case it is essential as far as the boundedness of the performance index is concerned.

2.3.3.1 Finite control horizon

The solution of problem 2.3.2 is very simple and some how obvious. In fact, according to it, the actual value of the control variable is made to depend on the optimal estimate of the state of the system (t.e. the state of the Kalman filter) through a gain matrix resulting from the minimization of the *deterministic* version of the performance index (2.3.34) in an LQ context, namely

$$J_d = \int_{t_0}^{t_f}(x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt$$

The precise statement of the relevant result is given in the following theorem

Theorem 2.3.7 Let $Z = 0$, t_0 and t_f be given such that $-\infty < t_0 < t_f < \infty$. Then the solution of Problem 2.3.2 is

$$u_c^o(\hat{x},t) = K(t)\hat{x} \tag{2.3.35}$$

where $\hat{x}$ is the state of Kalman filter

$$\dot{\hat{x}}(t) = [A(t) + L(t)C(t)]\hat{x}(t) + B(t)u_c^o(\hat{x}(t),t) - L(t)y(t) \tag{2.3.36}$$

with $\hat{x}(t_0) = \bar{x}_0$. In equations (2.3.35), (2.3.36), $K(t) = -R^{-1}(t)B'(t)P(t)$ and $L(t) = -\Pi(t)C'(t)W^{-1}$, where P and Π are the solutions (unique, symmetric and positive semidefinite) of the differential Riccati equations

$$\dot{P}(t) = -P(t)A(t) - A'(t)P(t) + P(t)B(t)R^{-1}(t)B'(t)P(t) - Q(t),$$

$$\dot{\Pi}(t) = \Pi(t)A'(t) + A(t)\Pi(t) - \Pi(t)C'(t)W^{-1}C(t)\Pi(t) + V$$

satisfying the boundary conditions $P(t_f) = 0$, $\Pi(t_0) = \Pi_0$, respectively.

Remark 2.3.8 *(Optimal value of the performance index)*
The value of J resulting from the implementation of the control low given in Theorem 2.3.7 can easily be evaluated by computing the quantity J_r which is defined in the proof of the quoted theorem . By taking into account equation (2.3.35) it follows that

$$J_r = tr\left[\int_{t_0}^{t_f} Q(t)\Pi(t)dt\right] + E\left[\int_{t_0}^{t_f} \hat{x}'(t)[Q(t) + K'(t)R(t)K(t)]\hat{x}(t)dt\right]$$

In view of Remark 2.2.9 the second term is given by

$$tr\left[P_K(t_0)\bar{x}_0 x_0' + \int_{t_0}^{t_f} L(t)WL'(t)P_K(t)dt\right]$$

where P_K is the solution of the equation

$$\dot{P}_K = -P_K(A + BK) - (A + BK)'P_K - (Q + K'RK)$$

with the boundary condition $P_K(t_f) = 0$. In writing down these relations the equation for $\hat{x}$, the relevant boundary condition and the circumstance that the noise intensity is W have been taken into consideration. It is straightforward to check that the DRE for P (in the statement of the quoted theorem) reduces to the differential equation given above, provided that the terms $\pm PBK$, $\pm K'B'P$ are added to the DRE itself and the expression of K is taken into account. Therefore it follows that

$$J_r = \bar{x}_0'P(t_0)\bar{x}_0 + tr\left[\int_{t_0}^{t_f} [Q(t)\Pi(t) + L(t)WL'(t)P(t)]dt\right].$$

2.3.3.2 Infinite control horizon

Unbounded control intervals can be dealt with also in a stochastic context and a partially nice solution found when the problem at hand is stationary, thus paralleling the results of the LQ and filtering framework. Consistent with eq.(2.3.34), the performance index to be minimized is

$$J = E\left[\lim_{\substack{t_0 \to -\infty \\ t_f \to \infty}} \frac{1}{t_f - t_0} \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt\right] \qquad (2.3.37)$$

where the matrices Q and R satisfy the usual continuity and sign definitions assumptions. Moreover, the system (2.3.1)-(2.3.6) is controllable and reconstructable for all t, the initial state is zero, $Z = 0$ and $W > 0$. If these assumptions are satisfied the solutions of the two DRE relevant to Problem 2.3.2 can indefinitely extended and the following theorem holds

Theorem 2.3.8 Assume that the above assumptions hold. Then the solution of Problem 2.3.2 when the control interval is unbounded that is when the performance index is given by equation (2.3.37) , is constituted by the control low

$$u_c^o(\hat{x},t) = \overline{K}(t)\hat{x}$$

where $\hat{x}$ is the state of the Kalman filter

$$\dot{\hat{x}}(t) = [A(t) + \overline{L}(t)C(t)]\hat{x}(t) + B(t)u_c^o(\hat{x}(t),t) - \overline{L}(t)y(t)$$

with $\overline{K}(t) = -R^{-1}(t)B'(t)\overline{P}(t)$ and $\overline{L}(t) = -\Pi(t)C'(t)W^{-1}$. The matrices $\overline{P}$ and $\overline{\Pi}$ are given by

$$\overline{P}(t) = \lim_{t_f \to \infty} P(t,t_f),$$

$$\overline{\Pi}(t) = \lim_{t_0 \to -\infty} \Pi(t,t_0),$$

where $P(t,t_f)$ and $\Pi(t,t_0)$ are the solutions of the differential Riccati equations specified in Theorem 2.3.7 with the boundary conditions $P(t_f,t_f) = 0$ and $\Pi(t_0,t_0) = 0$.

Remark 2.3.9 *(Optimal value of the performance index)*
If the LQG problem over an infinite interval admits a solution, the optimal value of the performance index can easily be evaluated by referring to Remark 2.3.8. Thus

$$J^o = \lim_{\substack{t_0 \to -\infty \\ t_f \to \infty}} \frac{1}{t_f - t_0} tr\left[\int_{t_0}^{t_f} [Q(t)\overline{\Pi}(t) + \overline{L}(t)W\overline{L}'(t)\overline{P}(t)]dt\right].$$

The case when all the problem data are constant deserves particular attention: the most significant features are the constancy of matrices $\overline{P}$ and $\overline{\Pi}$ (and hence of matrices $\overline{K}$ and $\overline{L}$, too) and the fact that they satisfy the ARE resulting from setting to zero the derivatives in the DRE of the statement of Theorem 2.3.8. The importance of this particular framework justifies the formal presentation of the relevant result in the forthcoming theorem where (unnecessarily) restrictive assumptions are made in order to simplify its statement and some results concerning Riccati equations are exploited.

Theorem 2.3.9 Let the matrices *A*, *B*, *C*, $Q := Q^{*}{}'Q^{*}$, R be constant, $Z = 0$, $V := V^{*}V^{*}{}'$, $W > 0$. Moreover let the couples (A,B) and (A,V^{*}) be reachable and the couples (A,C) and (A,Q^{*}) observable. Then the solution of Problem 2.3.2 when the control interval is not bounded, i.e. when the performance index is given by eq. (2.3.37), is specified by the control low

$$u_c^o(\hat{x}) = \overline{K}\hat{x}$$

where $\hat{x}$ is the state of the Kalman filter

$$\dot{\hat{x}}(t) = (A + \overline{L}C)\hat{x}(t) + Bu_c^o(\hat{x}(t)) - \overline{L}y(t)$$

with $\overline{K} = -R^{-1}B'\overline{P}$ and $\overline{L} = -\Pi C'W^{-1}$. The matrices $\overline{P}$ and $\overline{\Pi}$ are the unique, positive definite solutions of the algebraic Riccati equations

$$0 = PA + A'P - PBR^{-1}B'R + Q,$$

$$0 = \Pi A' + A\Pi - \Pi C' W^{-1} C\Pi + V \, .$$

Remark 2.3.10 (*Optimal value of the performance index in the time-invariant case*)
In view of Remark 2.3.9 , the optimal value of the performance index when the LQG problem is time-invariant and the control interval is unbounded is given simply by

$$J^o = tr[Q\overline{\Pi} + \overline{P}\overline{L}W\overline{L}']$$

This expression implies that $J^o \geq tr[Q\overline{\Pi}]$ since the second term is nonnegative. (The eigenvalues of the product of the two symmetric positive semidefinite matrices are nonnegative). This inequality holds independently of the actual matrix *R*. Therefore, even the control cost becomes negligible (i.e. when $R \to 0$), the value of the performance index cannot be less than $tr[Q\overline{\Pi}]$ which, in a sense, might be seen as the *price* to be paid because of the *imprecise* knowledge of the system state. Since it can also be proved that

$$J^o = tr[\overline{P}V + \overline{\Pi}\overline{K}'R\overline{K}] \, ,$$

the conclusion can be drown that $J^o \geq tr[\overline{P}V]$ and again, even when the output measurement becomes arbitrary accurate $(W \to 0)$, the optimal value of the performance index cannot be less than $tr[\overline{P}V]$ which, in a sense, might be seen as the *price* to be paid because of the *presence* of the input noise.

Remark 2.3.11 (*Stability of the LQG solution*)
When the assumptions of Theorem 2.3.10 hold, the resulting control system is asymptotically stable. Indeed, since the Kalman filter, which is the core of the controller, has the structure of a state observer. It follows that the eigenvalues of the control system are those of matrices $A + B\overline{K}$ and $A + \overline{L}C$. All these eigenvalues have negative real parts because the solutions of the ARE (from which the matrices $\overline{K}$ and $\overline{L}$ originate) are stabilizing (recall the assumptions of Theorem 2.3.10).
The solution of the optimal regulator problem has been proved *robust* in terms of phase and gain margins (see Subsection 2.2.4.2 of Section 2.2.4 in Chapter 2.2.1) The same conclusions hold in the filtering context because of the duality between the two problems. Thus one might conclude that the controller defined in Theorem 2.3.9 implies that the resulting control system is endowed with analogous robustness properties with regard to the presence of phase and gain uncertainties on the *control side* and/or the *output side*. This actually feels to be true.
The unpleasant output is caused by the following fact. The transfer function

$$T_u(s) := -\overline{K}(sI - (A + B\overline{K} + \overline{L}C))^{-1}\overline{L}G(s)$$

does not coincide with the transfer function $T_c(s) := \overline{K}(sI - A)^{-1}B$ which is expedient in proving the robustness of the solution of the LQ problem. A similar discussion applies to the other side of $G(s)$, with reference to the transfer function

$$T_y(s) := -G(s)\overline{K}(sI - (A + B\overline{K} + \overline{L}C))^{-1}\overline{L}$$

and to the transfer function $T_f(s) := C(sI - A)^{-1}\overline{L}$ which , in the Kalman filter framework, plays the same role, from the robusthess point of view, as T_c does in the optimal regulator setting. It is steel easy to check that T_f is the transfer function which results from cutting the above quoted scheme at the point P_f. Therefore, if the four matrices *Q*, *R*, *V*, *W*, are *given* data of the problem and the available knowledge on the controlled process is not accurate, no robustness properties can *a priori* be guaranteed to the control system either on the actuator or on sensor sides. On the other hand, if, as often is the case, the four matrices above are to be meant as *free* parameters to be selected while carrying over a sequence of trials suggested by a synthetic procedure which exploits the LQG results, then a wise choice of them may again ensure specific robustness properties. In fact, by resorting to reasoning similar to which led to Theorem 2.2.12 of Subsection 2.2.4.3, the following results can be proved. They are started under the assumptions that the number of control variables u equals the number of output variables y and the matrices B C are full rank.

Theorem 2.3.11 Let the triple (A, B, C) be minimal and $V = vBB'$. Then, if no transmission zeros of the triple (A, B, C) has positive real part, the function T_y approaches the function T_f as $q \to \infty$.

Remark 2.3.12 *(Alternative statement of Theorem 2.3.10 and 2.3.11)*
By recalling the role plaid by the matrices Q and R in specifying the meaning of the performance index and by matrices V and W in defining the noises characteristics, it should be fairly obvious that instead letting matrices Q and V go to infinity, we could let matrices R and W go to zero.

Chapter 3

Robust optimal control

3.1 H_∞ Optimal Control: Riccati-Approach

3.1.1 Introduction

In the last chapter, we considered the problem with respect to the H_2 norm. The performance specifications were given in the time domain. For single input single output (SISO) problems, for specifications in frequency domain the H_∞ norm is an adequate tool. In this way we are naturally lead to the question of how controllers can be characterized in a way which minimizes the closed loop transfer function F_{zw} with respect to the H_∞ norm. There are two important methods for solving this problem. One is based on two Riccati equations similar to those used in the H_2 problem. It will be analyzed in this chapter, whereas the other method uses linear matrix inequalities.
We are lead to the characterization of suboptimal controllers instead of optimal controllers. The basic idea for solving the characterization problem is, as for the H_2 problem a change of variables of the kind $v = u - Fx$, with a matrix F with is related to a Riccati equation. The resulting problem is again an output estimation problem, which can be reduced in several steps to a full information problem. The technical details are much more complicated as for the H_2 problem. Thereby the basic structure of the plant is similar to that in H_2 control. In particular, the assumption $D_{11} = 0$ is made again. This assumption is natural for H_2 problems but restrictive for H_∞ problems, since the introduction of weights in order to get a certain closed loop frequency response normally leads to generalized plants with $D_{11} \neq 0$. It is possible to reduce problems with $D_{11} \neq 0$ to ones with $D_{11} = 0$ by a procedure called loop shifting. The introduction of frequency dependent weights leads in many situations to a so-called mixed sensitivity problem. We discuss it together with problems based on other weighting schemes and finally present a result on pole-zero cancellations.

3.1.2 Formulation of the general H_∞ problem

We start with a general plant of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

i.e.

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right].$$

A controller $K(s)$ is denoted as admissible, if it is proper and if it stabilizes internally the system $F_{zw}(s)$. We now formulate the following problem.

Problem 3.1.1 (*Optimal H_∞ problem*) Find all admissible controllers $K(s)$ which minimize the H_∞ norm of the feedback system. i.e. all admissible controllers that minimize $\|F_{zw}\|_\infty$.

For the minimization of the H_∞ norm it is more natural to ask for all suboptimal controllers. Finding an optimal controller is more difficult and, besides this, optimal controllers for the H_∞ problem are not unique. They can be viewed as the limit case for suboptimal controllers and are not explicitly constructed. Therefore we are led to the following problem.

Problem 3.1.2 (*Suboptimal H_∞ problem*) For a given $\gamma > 0$, find all admissible controllers $K(s)$ with $\|F_{zw}\|_\infty < \gamma$. Such a controller is denoted as ***suboptimal***.

We define

$$\gamma_{opt} = \inf\{\|F_{zw}\|_\infty \mid K(s) is admissable\}.$$

Note that for $\gamma = \gamma_{opt}$, there are no suboptimal controllers. For numbers γ, which are greater than the optimal value γ_{opt}, there are always admissible controllers with $\|F_{zw}\|_\infty < \gamma$. It is possible to characterize the suboptimal controllers belonging to $\gamma > \gamma_{opt}$ completely.

3.1.3 Characterization of H_∞ suboptimal controllers by means of Riccati equations

3.1.3.1 Characterization theorem for output feedback

In this section, we describe suboptimal H_∞ controllers for problems with a special structure. The following assumptions are made.

(a1) (A, B_1) is stabilizable and (C_1, A) is detectable.
(a2) (A, B_2) is stabilizable and (C_2, A) is detectable.
(a3) $D_{12}'C_1 = 0$ and $D_{12}'D_{12} = I$.
(a4) $D_{11} = 0$ and $D_{22} = 0$.

These assumptions are too restrictive. Later it will be shown how they can be relaxed.
Our first question is under which conditions internal stability is equivalent to

$$F_{zw} \in \Re H_\infty.$$

Corollary 3.1.3.1 The assumptions (a1),(a3),(a4) imply that the feedback loop is internally stable if and only if $F_{zw} \in \Re H_\infty$.

For the next theorem, the following Hamiltonian matrices are used:

$$H_\infty = \begin{bmatrix} A & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}$$

$$J_\infty = \begin{bmatrix} A' & \gamma^{-2}C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}.$$

Theorem 3.1.3.1 Suppose the assumptions (a1)-(a5) hold. Then there exists an admissible controller with $\|F_{zw}\|_\infty < \gamma$ is and only if the following conditions are fulfilled:

(i) $H_\infty \in dom(Ric)$ and $X_\infty = Ric(H_\infty) \geq 0$;

(ii) $J_\infty \in dom(Ric)$ and $Y_\infty = Ric(J_\infty) \geq 0$;

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

If these conditions hold, such a controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

with

$$\hat{A}_\infty A + \gamma^{-2}B_1B_1'X_\infty + Z_\infty L_\infty C_2$$

$$F_\infty = -B_2'X_\infty, \qquad L_\infty = -Y_\infty C_2', \qquad Z_\infty = (I - \gamma^2 Y_\infty X_\infty)^{-1}$$

It is possible to describe this controller with an observer. The controller can equivalently be written in the form

$$\dot{\tilde{x}} = A\tilde{x} + B_1\tilde{w}_{worst} + B_2u + Z_\infty L_\infty (C_2\tilde{x} - y)$$

$$u = F_\infty \tilde{x}, \quad \tilde{w}_{worst} = \gamma^{-2}B_1'X_\infty \tilde{x}.$$

The first equation defines an observer. The term $\tilde{w}_{worst} = \gamma^{-2}B_1'X_\infty\tilde{x}$ can be understood as an estimate of the disturbance $w_{worst} = \gamma^{-2}B_1'X_\infty x$. In this way, one gets a controller-observer structure similar to that for the H_2 problem. In contrast to this problem, the vector B_1 enters in the H_∞ observer.

The H_∞ suboptimal controller has also the representation

$$K_{sub} = (s) = -Z_\infty L_\infty (sI - \hat{A}_\infty)^{-1} F_\infty.$$

It has as many states as the generalized plant $P(s)$ and is strictly proper. The Riccati equations for X_∞ and Y_∞ are

$$X_\infty A + A'X_\infty - X_\infty(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty + C_1C_1' = 0 \tag{3.1.1}$$

$$AY_\infty + Y_\infty A' - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty + B_1B_1' = 0 \tag{3.1.2}$$

3.1.3.2 Outline of the proof

It is seen that the inequality $\|F_{zw}\|_\infty < \gamma$ can equivalently be written as

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 < 0 \qquad \text{for all} \qquad w \in L_2[0,\infty), \quad w \neq 0 \tag{3.1.3}$$

Here *z* is given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_1 u, \qquad x(0) = 0 \\ z &= C_1 x + D_{12} u \end{aligned},$$

where *u* is the controller output. We assume that the Riccati equation (3.1.1) has a solution X_∞. Using this equation and assumption (a3) and supposing that $x(t)$ tends to $x_\infty = 0$ as $t \to 0$ one obtains

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 = \|u + B_2' X_\infty x\|_2^2 - \gamma^2\left\|w - \gamma^{-2} B_1' X_\infty x\right\|_2^2 \tag{3.1.4}$$

If all states are available, the choice

$$u = -B_2' Z_\infty x \tag{3.1.5}$$

can be made. This leads to

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 = -\gamma^2\left\|w - \gamma^{-2} B_1' X_\infty x\right\|_2^2 \qquad \text{for every} \quad w \in L_2[0,\infty).$$

The difference on the right-hand side vanishes only for $x = 0$ and $w = 0$. Hence, with the controller (3.1.5) inequality (3.1.3) holds and therefore we have $\|F_{zw}\|_\infty < \gamma$.

We need the following matrices

$$A_{F_\infty} = A + B_2 F_\infty, \; C_{1F_\infty} = C_1 + D_{12} F_\infty.$$

The next lemma is a first step in solving the full information (FI) problem for H_∞ optimal control.

Lemma 3.1.3.1 Suppose $H_\infty \in dom(Ric)$ and $X_\infty = Ric(H_\infty) \geq 0$. Then the inequality $\|F_{zw}\|_\infty < \gamma$ is fulfilled if the controller is given by the constant matrix

$$K(s) = -B_2' X_\infty.$$

This lemma is also the key for the solution of the observable eigenvalue (OE) H_∞ problem.

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